Critical buckling load analysis of Euler-Bernoulli beam on two-parameter foundations using Galerkin method

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Abstract. The critical buckling load determination of Euler-Bernoulli beams on two-parameter elastic foundations (EBBo2PFs) is important to avert buckling failures. The governing equation for buckling of thin beam on two-parameter elastic foundation is a homogeneous ordinary differential equation (HODE) of fourth order and constant parameters when the beam is prismatic and homogeneous. The HODE has been solved in this work by Galerkin method for simply supported, clamped and clamped-simply supported ends. One-parameter algebraic shape function formulation was used to reduce the problem to an algebraic eigenvalue problem, which is solved to find the critical buckling load for each studied case. The critical buckling load for EBBo2PF for simply supported boundary conditions was found to be closely identical to the exact solutions. The solutions for clamped-clamped edges and clamped-simple supports were found to be accurate. The merit of the Galerkin method is the simplicity and the accuracy even when one-parameter shape function has been used.

Keywords: Galerkin variational method, Euler-Bernoulli beam, two-parameter elastic foundation, critical buckling load, shape function.

Nomenclature

x	Longitudinal coordinate axis
l	Length of the beam
$w(x)$	Deflection
$r(x)$	Foundation reaction on the beam
k	Winkler modulus
k_1, k_2	Two-parameters of a two-parameter elastic foundation
E	Young's modulus
I	Moment of inertia
P	Compressive load
$q(x)$	Applied transverse load
$\varphi_i(x)$	Shape function
c_i	Generalized parameter of displacement function
β	Parameter defined in terms of P and EI
α_1	Parameter defined in terms of k_1 and EI
α_2	Parameter defined in terms of k_2 and EI
$[K]$	Stability matrix
$\{C\}$	Matrix of generalized deflection parameters
k_{ij}	Element of the stiffness matrix
a_0, a_1, a_2, a_3, a_4	Polynomial constants used to define the deflection function
I_1	Integral defined in terms of the integrand $\varphi_1^{iv}\varphi_1$
I_2	Integral defined in terms of the integrand $\varphi''_1\varphi_1$
I_2	Integral defined in terms of the integrand $\varphi_1^2(x)$
γ_1	Vertical distribution parameter

EBBo2PF(s)	Euler-Bernoulli beam on two-parameter foundation(s)
HODE	Homogeneous ordinary differential equation
NODE	Nonhomogeneous ordinary differential equation
EBBT	Euler-Bernoulli beam theory
TBT	Timoshenko beam theory
DTM	Differential transform method
FEM	Finite element method
DEoB	Differential equations of buckling
RDM	Recursive differentiation method
EBBoEF(s)	Euler-Bernoulli beam on elastic foundation(s)
VIM	Variational iteration method
EBBoWF	Euler-Bernoulli beam on Winkler foundation
FSTM	Finite sine transform method
GITM	Generalized integral transform method
BVP	Boundary value problem
PCM	Point collocation method
SVIM	Stodola-Vianello iteration method
LSWRM	Least square weighted residual method
GVF	Galerkin variational functional
FSM	Fourier series method
$F_1(\alpha_1 l^4, \alpha_2 l^2)$	Critical buckling load parameter for simply supported Ebbo2pf
$F_2(\alpha_1 l^4, \alpha_2 l^2)$	Critical buckling load parameter for clamped-clamped Ebbo2pf
$F_3(\alpha_1 l^4, \alpha_2 l^2)$	Critical buckling load parameter for clamped-simply supported Ebbo2pf

1. Introduction

The differential equations of elastic stability of beams resting on elastic foundations have been derived by incorporating the reactive forces of the elastic foundation in the equation of beam stability. Beam models commonly used depend on the depth to thickness ratios. When the beam depth to thickness ratio is less than 0.05, the beam is called thin or slender beam. Euler-Bernoulli beam theory (EBBT) is used for thin beam because the Euler-Bernoulli-Navier orthogonality hypothesis used in the formulation disregards shear strain that play significant roles in the behaviour of moderately thick and thick beams. Timoshenko beam theory (TBT), shear deformable beam theories, and refined beam theories formulated by several researchers such as Dahake and Ghugal [1], Levinson [2], Sayyad and Ghugal [3], Ghugal and Shimpi [4], Yue [5], are used for moderately thick and thick beams where the depth to span ratios exceed 0.05.

This work assumes a depth to span ratio of beams to be less than 0.05, and hence uses EBBT.

Elastic foundation models have been derived based on discrete parameter and continuously distributed parameter assumptions. Discrete parameter models use discretization of the foundations at discrete points, and hence a limited number of parameters to describe the foundation. However, continuously distributed parameter models derive the foundation reactions using the mathematical theory of elasticity resulting in complex differential equations that are difficult to solve in analytical form. The inherent simplicity of the resulting mathematical representation of discrete parameter foundations make them more frequently used. Discrete parameter foundation models include (i) Winkler foundation model (ii) Pasternak, Vlasov, Hetenyi and Filonenko-Borodich foundations, and (iii) Kerr foundation.

Winkler foundation model which is shown in Figure 1 is a one-parameter foundation model. The foundation is represented as a system of vertical, closely spaced, non-interacting linear elastic springs whose vertical stiffness is directly proportional to the vertical deflection of the point considered [6-8]. In the Winkler foundation, the foundation reaction $r(x)$ (at any point) is directly proportional to the beam deflection $(w(x))$ at the point and the constant of proportionality is the Winkler foundation constant, k, which is the one-parameter used to characterise the soil.

The equation for the foundation reaction ($r(x)$) for the Winkler model is given [6-8]:

$$r(x) = kw(x).$$

The main disadvantage of the Winkler foundation model is its failure to consider the shear interaction of the vertical springs and the resulting discontinuity of displacements caused.

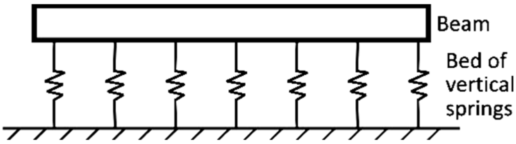

Fig. 1. Thin beam resting on a bed of vertical linear elastic springs

Two-parameter discrete foundation models were derived by Pasternak, Vlasov, Hetenyi and Filonenko-Borodich in order to overcome the displacement discontinuity issues in the one-parameter Winkler foundation model. In the two-parameter discrete foundation models, the foundation is represented using a bed of vertical, closely spaced, interacting linear elastic springs. Shear interactions are introduced between the adjoining vertical springs by the use of shear layer and coupling. The foundation is thus modelled using two-parameters; the first parameter, denoted by k_1 is the stiffness in the vertical direction, and is similar to the Winkler foundation parameter. The second parameter, k_2, is the shear interaction effect of the vertical springs [9-11]. An illustration of two-parameter discrete foundation models is shown in Fig. 2, which presents it as a bed of closely spaced, vertical, interacting springs with adjoining springs linked together using shear coupling.

The foundation reaction $r(x)$ for two-parameter models is expressed by [9], [11]:

$$r(x) = k_1 w(x) - k_2 \frac{d^2 w(x)}{dx^2}.$$

Kerr [12-14] presented a three-parameter discrete foundation model, which is rarely used.

Vlasov and Leontiev [15] used an energy minimization technique to derive the two-parameter Vlasov-Leontiev simplified elastic continuum foundation model. The model introduced an arbitrary parameter γ_1 to define the vertical distribution of soil deformation.

Jones and Xenophontoss [16] derived an expression for the vertical distribution parameter γ_1 in terms of the displacement characteristics, but failed to derive a method for finding γ_1. In further studies, Vallabhan and Das [17, 18], found the γ_1 using an iterative procedure.

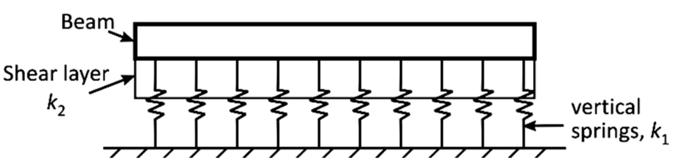

Fig. 2. Beam supported on two-parameter discrete foundation graphically illustrated as a bed of vertical closely spaced, interacting linear elastic (Hookean) springs

2. Literature review

Theory of elasticity methods were used for beam on elastic foundation and analysis by Akhazhanov et al. [19-22], Huang et al. [23], Akhmedeev et al. [24], Zhang et al. [25], and Gbolami and Alizadeh [26].

Gulkan and Alemdar [27] and Teodoru and Muscat [28] used the finite element method (FEM) to study beams on two-parameter elastic foundations. FEMs were also used for EBBoEF by Alzubaidi et al. [29], Wieckowski and Swiatkiewiez [30], and Worku and Habte [31].

Olotu et al. [32] used the differential transform method (DTM) to find numerical solutions for natural frequencies of non prismatic beams supported on variable one-parameter foundations. The DTM reduced the governing equation to an algebraic problem which was solved using computer algorithms for various boundary conditions. They however failed to solve buckling analysis of thin beams on elastic foundations using DTM.

Aslami and Akinov [33] presented closed form solutions of EBBo2PFs by reducing the governing equations to a system of first order differential equations, which were subsequently solved using Jordan method.

Hetenyi [34], Timoshenko and Gere [35], Wang et al. [36] have derived differential equations of buckling (DEoB) of Euler-Bernoulli beams on elastic foundations (EBBoEFs). They also derived analytical solutions to the DEoB for various boundary conditions.

Taha and Hadima [37] and Taha [38] used Recursive differentiation method (RDM) on the DEoB to obtain analytical solutions for the critical buckling loads of non uniform EBBoEFs. Soltani [39] presented a finite element method to solve the DEoB for EBBoEF. Aristizabal-Ochoa [40] and Hassan [41] have derived solutions for EBBoEFs under various boundary conditions. Anghel and Mares [42] used the method of collocation to solve the stability problems of EBBoEFs. Atay and Coskun [43] applied the Variational iteration method (VIM) to the analysis of the stability of EBBoEF for prismatic and non-prismatic cross-sectional beam geometries; and for variety of boundary conditions. Akgoz et al. [44] investigated the bending analysis of beams on elastic foundations but failed to study buckling analysis.

Ike [6] used the finite sine transform method (FSTM) to simplify the governing equation of free vibration of simply supported EBBoWF to an integral equation, and ultimately to an algebraic problem. The method was found suited for EBBoWF with Dirichlet boundary conditions because the sinusoidal function which is the integral kernel of the FSTM satisfies the geometric and force boundary conditions. The FSTM gave exact eigenvalues, but the work was not extended to buckling analysis.

Ike [7] utilized a point collocation method (PCM) to obtain approximate solution to the differential equation of flexure (DEoF) of Euler-Bernoulli beam resting on Winkler foundation (EBBoWF). In the PCM, the solutions were obtained in an approximate way only at the collocation points. Acceptable bending solutions that were comparable to previous solutions found in literature sources were obtained. However, the study failed to consider buckling analysis.

Ike [8] used the generalized integral transform method (GITM) to solve the free vibration problems of EBBoWF under various boundary conditions. In the GITM, the eigenfrequencies of free vibrations of thin beams with equivalent boundary conditions were used as kernel functions to formulate the boundary value problem (BVP) as an integral equation. The kernel functions used were exact shape functions, and the resulting eigenvalues were exact. However, the work did not consider bucking analysis.

Ike [9] used SVIM and sinusoidal shape functions to derive exact buckling loads of EBBo2PFs. Ike et al. [10] and Ike [11] utilized SVIM for buckling solutions of EBBo2PFs using polynomial shape function for beams with both ends clamped and beams with simply supported ends, respectively.

Ike et al. [45] used Picard's successive iteration method to solve Euler buckling problem with pinned ends. Ofondu et al. [46] applied Stodola-Vianello iteration method (SVIM) to find solutions to Euler column buckling problems with one end clamped and the other on pin support. Ikwueze et al. [47] utilized Least squares weighted residual method (LSWRM) for the critical buckling load analysis of Euler columns with one end fixed and the other end pinned. Mama et al. [48] used fifth degree polynomials as shape functions in a FEM to obtain accurate critical buckling load solutions of EBBoWF.

Ike et al. [49] used the SVIM and polynomial functions that satisfy the boundary conditions of EBBoWF with both ends clamped to find satisfactory solutions for critical buckling loads. In another study, Ike [50] used the SVIM and polynomial functions that satisfy the simply supported boundary conditions to find approximate critical buckling load solutions for simply supported

EBBoWF.

Ike [51] used SVIM and exact sinusoidal shape functions to obtain exact buckling solutions to simply supported EBBoWF.

Ike [52] used Fourier series method (FSM) to derive analytical solutions to simply supported EBBo2PF. The FSM has the advantage that the sine functions used in the series are orthogonal functions that are readily differentiated. The problem simplified by orthogonalization of the series to an algebraic eigenvalue problem which was solved to obtain exact eigenvalues.

Ike [53] used Ritz variational method to derive accurate buckling load solutions of EBBo2PFs for both ends simply supported, and both ends clamped and for one end simply supported and the other end clamped. The work developed suitable polynomial shape functions that satisfied the boundary conditions considered and used them as shape functions in deriving the Ritz variational functions for minimization with respect to the undetermined parameters. Naidu and Rao [54] presented buckling solutions for uniform beams resting on two-parameter elastic foundations. Rao and Raju [55] developed analytical solutions for the vibration and buckling analysis of thin beams resting on Pasternak foundations for clamped boundary conditions.

In this work, polynomial basis functions are used in the Galerkin method to obtain approximate critical buckling load solutions of EBBo2PFs with:

(i) simply supported ends,

(ii) clamped ends, and,

(iii) one end clamped and one end simply supported.

3. Governing equation

In general, the differential equation for the buckling of Euler-Bernoulli beam resting on two-parameter foundation (EBBo2PF) is the fourth order nonhomogeneous ordinary differential equation (NODE) [50, 52, 53]:

$$\frac{d^2}{dx^2}\left(EI\frac{d^2w}{dx^2}\right)+P\frac{d^2w}{dx^2}+k_1w-k_2\frac{d^2w}{dx^2}=q(x), \quad 0<x\leq l, \tag{1}$$

where x is the longitudinal axis of the beam, E is the Young's modulus, I is the moment of inertia, P is the axial force, $q(x)$ is the applied transverse load intensity, k_1 is the first modulus/parameter of the foundation, k_2 is the second modulus/parameter of the foundation.

When there is no applied transverse load on the EBBo2PF, $q(x)=0$, and Eq. (1) simplifies for homogeneous beams with prismatic cross-sections to the homogeneous ordinary differential equation (HODE) with constant coefficients given by Eq. (2) [50, 52, 53]:

$$EI\frac{d^4w}{dx^4}+P\frac{d^2w}{dx^2}+k_1w-k_2\frac{d^2w}{dx^2}=0. \tag{2}$$

Dividing Eq. (2) by EI gives Eq. (3):

$$\frac{d^4w}{dx^4}+\frac{P}{EI}\frac{d^2w}{dx^2}+\frac{k_1}{EI}w-\frac{k_2}{EI}\frac{d^2w}{dx^2}=0. \tag{3}$$

Alternatively, Eq. (3) can be expressed as Eq. (4):

$$\frac{d^4w}{dx^4}+(\beta-\alpha_2)\frac{d^2w}{dx^2}+\alpha_1w=0, \tag{4}$$

where:

$$\beta = \frac{P}{EI}, \quad \alpha_1 = \frac{k_1}{EI}, \quad \alpha_2 = \frac{k_2}{EI}. \tag{5}$$

4. Methodology

4.1. Galerkin variational functional (GVF)

The deflection function $w(x)$ is expressed in terms of a finite number of linear combinations of shape functions $\varphi_i(x)$ that satisfy the boundary conditions of the beam as:

$$w(x) = \sum_{i=1}^{n} c_i \varphi_i(x), \tag{6}$$

where c_i are the generalized parameters of the deflection function $w(x)$.

The Galerkin formulation of the governing equation then becomes expressed as Eq. (7):

$$\sum_{I=1}^{n} c_i \int_0^l \left(\varphi_i^{iv}(x) + (\beta - \alpha_2)\varphi''_i(x) + \alpha_1 \varphi_i(x)\right) \varphi_j(x)\, dx = 0. \tag{7}$$

Expanding Eq. (7) gives:

$$[K]\{C\} = 0, \tag{8}$$

where $[K]$ is the stability matrix, $\{C\}$ is the matrix of generalized parameters.

The ij element of $[K]$ is denoted as k_{ij} and found as:

$$k_{ij} = \int_0^l \left(\varphi_i^{iv}(x) + (\beta - \alpha_2)\varphi''_i(x) + \alpha_1 \varphi_i(x)\right)\varphi_j(x)\, dx. \tag{9}$$

4.2. Shape functions for various boundary conditions

4.2.1. Case 1: simply supported EBBo2PF

For simple supports at $x = 0$ and $x = l$ as shown in Fig. 3, the boundary conditions are:

$$\begin{aligned} w(0) = w''(0) = 0, \\ w(l) = w''(l) = 0. \end{aligned} \tag{10}$$

Let:

$$w(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4, \tag{11}$$

where a_0, a_1, a_2, a_3 and a_4 are the polynomial constants.

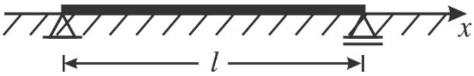

Fig. 3. Simply supported EBBo2PF

By differentiation:

$$w'(x) = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3, \tag{12a}$$

$$w''(x) = 2a_2 + 6a_3x + 12a_4x^2. \tag{12b}$$

Applying the boundary conditions:

$$w(0) = a_0 = 0, \tag{12c}$$
$$w''(0) = 2a_2 = 0, \tag{12d}$$
$$w(l) = a_1l + a_3l^3 + a_4l^4 = 0, \tag{12e}$$
$$w''(l) = 6a_3l + 12a_4l^2 = 0. \tag{12f}$$

From Eq. (12f):

$$a_3 = -2a_4l. \tag{13}$$

Then:

$$\begin{gathered} a_1l = -a_4l^4 - a_3l^3 = -a_4l^4 + 2a_4l^4 = a_4l^4, \\ a_1 = a_4l^3. \end{gathered} \tag{14}$$

Hence:

$$w(x) = a_4(l^3x - 2lx^3 + x^4). \tag{15}$$

Hence a one-parameter shape function for this case of simple supports at the ends is given by Eq. (16):

$$\varphi_1(x) = x^4 - 2lx^3 + l^3x. \tag{16}$$

4.2.2. Case 2: clamped-clamped EBBo2PF

EBBo2PF clamped at $x = 0$, and $x = l$, as shown in Fig. 4 is considered.

Fig. 4. EBBo2PF with clamped edges

The boundary conditions are:

$$\begin{gathered} w(0) = w'(0) = 0, \\ w(l) = w'(l) = 0. \end{gathered} \tag{17}$$

Using the fourth degree polynomial in Eq. (11), the application of boundary conditions yield:

$$w(0) = a_0 = 0,$$
$$w'(0) = a_1 = 0,$$
$$w(l) = a_2l^2 + a_3l^3 + a_4l^4 = 0,$$
$$w'(l) = 2a_2l + 3a_3l^2 + 4a_4l^3 = 0.$$

Solving simultaneously:

$$a_3 = -2a_4l,$$
$$a_2 = a_4l^2.$$

Assuming a quartic polynomial, a one-parameter deflection function can be derived using

Eq. (17) as:

$$w(x) = a_4(x^4 - 2lx^3 + l^2x^2). \tag{18}$$

Then for this case:

$$\varphi_1(x) = x^4 - 2lx^3 + l^2x^2. \tag{19}$$

4.2.3. Case 3: clamped – simply supported EBBo2PF

EBBo2PF clamped at $x = 0$, and simply supported at $x = l$ as shown in Fig. 5 is considered.

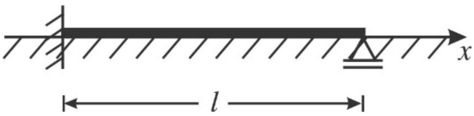

Fig. 5. EBBo2PF clamped at $x = 0$ and simply supported at $x = l$

The boundary conditions are:

$$\begin{aligned} &w(0) = w'(0) = 0, \\ &w(l) = w''(l) = 0. \end{aligned} \tag{20}$$

Using the fourth degree polynomial in Eq. (11), the boundary conditions yield:

$$\begin{aligned} &w(0) = a_0 = 0, \\ &w'(0) = a_1 = 0, \\ &w(l) = a_2l^2 + a_3l^3 + a_4l^4 = 0, \\ &w''(l) = 2a_2 + 6a_3l + 12a_4l^2 = 0. \end{aligned}$$

Solving:

$$\begin{aligned} &a_2 = 1.5a_4l^2, \\ &a_3 = -2.5a_4l. \end{aligned}$$

Using the boundary conditions, a one-parameter deflection function can be derived using Eq. (20) as:

$$w(x) = a_4(x^4 - 2.5lx^3 + 1.5l^2x^2). \tag{21}$$

Hence:

$$\varphi_1(x) = x^4 - 2.5lx^3 + 1.5l^2x^2. \tag{22}$$

4.3. One-Parameter GVF

Here:

$$w(x) = c_1\varphi_1(x). \tag{23}$$

Then the GVF is:

$$\int_0^l \left(c_1\varphi_1^{iv}(x) + (\beta - \alpha_2)c_1\varphi''_1(x) + c_1\alpha''_1\varphi_1(x)\right)\varphi_1(x)\,dx = 0. \tag{24}$$

Eq. (24) is simplified by factoring out c_1 to give:

$$c_1\left\{\int_0^l \left(\varphi_1^{iv}(x)\varphi_1(x) + (\beta - \alpha_2)\varphi''_1(x)\varphi_1(x) + \alpha_1\varphi_1^2(x)\right)dx\right\} = 0. \tag{24a}$$

For non-trivial solution, $c_1 \neq 0$.
Hence, the buckling equation is:

$$\int_0^l \left(\varphi_1^{iv}(x)\varphi_1(x) + (\beta - \alpha_2)\varphi''_1(x)\varphi_1(x) + \alpha_1\varphi_1^2(x)\right)dx = 0. \tag{24b}$$

Simplifying:

$$I_1 + (\beta - \alpha_2)I_2 + \alpha_1 I_3 = 0, \tag{24c}$$

where:

$$I_1 = \int_0^l \varphi_1^{iv}(x)\varphi_1(x)\,dx, \quad I_2 = \int_0^l \varphi''_1(x)\varphi_1(x)\,dx, \quad I_3 = \int_0^l \varphi_1^2(x)\,dx. \tag{24d}$$

Solving the algebraic equation:

$$\beta = \frac{P}{EI} = \alpha_2 - \left(\frac{I_1 + \alpha_1 I_3}{I_2}\right). \tag{24e}$$

Hence:

$$P = \frac{EI}{l^2}\left(\alpha_2 - \left(\frac{I_1 + \alpha_1 I_3}{I_2}\right)\right)l^2. \tag{24f}$$

5. Results

5.1. EBBo2PF with simply supported edges

In this case:

$$I_1 = 4.8l^5, \quad I_2 = -\frac{204l^7}{420}, \quad I_3 = -\frac{31l^9}{630}. \tag{25}$$

Then:

$$P = \frac{EI}{l^2}\left(\alpha_2 l^2 + \left(4.8l^7 + \frac{31\alpha_1 l^{11}}{630}\right)\frac{420}{204l^7}\right). \tag{26}$$

Simplifying:

$$P=\frac{EI}{l^2}(\alpha_2 l^2+9.882352941+0.101307189\alpha_1 l^4). \tag{27}$$

Eq. (27) can be expressed in the form:

$$P=\frac{EI}{l^2}F_1(\alpha_1 l^4,\alpha_2 l^2). \tag{28}$$

$F_1(\alpha_1 l^4,\alpha_2 l^2)$ is the buckling parameter. Approximately, F_1 can be expressed as:

$$F_1(\alpha_1 l^4,\alpha_2 l^2)=\alpha_2 l^2+9.88235+0.1013072\alpha_1 l^4. \tag{29}$$

$F_1(\alpha_1 l^4,\alpha_2 l^2)$ is calculated and presented in Table 1 for $\alpha_1 l^4=0$, 1, 100, 1900, 10000 and $\alpha_2 l^2=0$, $0.5\pi^2$, π^2, $2.5\pi^2$. Table 1 also presents previous values of F_1 calculated using FEM by Naidu and Rao [54] and Ike [52, 53] using Stodola-Vianello iteration method.

5.2. EBBo2PF with clamped edges

Here:

$$I_1=0.8l^5,\quad I_2=-\frac{2l^7}{105},\quad I_3=\frac{l^9}{630}. \tag{30}$$

Then:

$$P=\frac{EI}{l^2}\left(\alpha_2 l^2+\frac{105}{2l^7}\left(0.8l^5+\frac{\alpha_1 l^9}{630}\right)l^2\right). \tag{31}$$

Simplifying:

$$P=\frac{EI}{l^2}\left(\alpha_2 l^2+42+\frac{\alpha_1 l^4}{12}\right). \tag{32}$$

Eq. (32) can be expressed as:

$$P=\frac{EI}{l^2}F_2(\alpha_1 l^4,\alpha_2 l^2), \tag{33}$$

where:

$$F_2(\alpha_1 l^4,\alpha_2 l^2)=\alpha_2 l^2+42+\frac{\alpha_1 l^4}{12}. \tag{34}$$

$F_2(\alpha_1 l^4,\alpha_2 l^2)$ is calculated for $\alpha_1 l^4=0$, 1, 100 and $\alpha_2 l^2=0$, $0.5\pi^2$, π^2, $2.5\pi^2$ using Eq. (34) and presented in Table 2 together with previous results by Naidu and Rao [54] and Rao and Raju [55].

5.3. Clamped-Pinned EBBo2PF

Here:

$$I_1=1.8l^5,\quad I_2=-\frac{3l^7}{35},\quad I_3=\frac{19l^9}{2520}, \tag{35}$$

$$P = \frac{EI}{l^2}\left(\alpha_2 l^2 + \frac{35}{3l^7}\left(1.8l^5 + \frac{19\alpha_1 l^9}{2520}\right)l^2\right). \quad (36)$$

Simplifying:

$$P = \frac{EI}{l^2}(\alpha_2 l^2 + 21 + 0.087962962\alpha_1 l^4). \quad (37)$$

Eq. (37) can be expressed as:

$$P = \frac{EI}{l^2}F_3(\alpha_1 l^4, \alpha_2 l^2). \quad (38)$$

Approximating:

$$F_3(\alpha_1 l^4, \alpha_2 l^2) = \alpha_2 l^2 + 21 + 0.087963\alpha_1 l^4. \quad (39)$$

$F_3(\alpha_1 l^4, \alpha_2 l^2)$ is calculated for $\alpha_1 l^4 = 0$, 50, 100, $\alpha_2 l^2 = 0$, $0.5\pi^2$, π^2, $2.5\pi^2$ and presented in Table 3. Table 3 also presents previous results for EBBoWF (for $\alpha_2 l^2 = 0$) using variational iteration method (VIM) by Atay and Coskun [43] and exact solution by Wang et al. [36].

Table 1. Critical buckling load parameters of a simply supported Euler-Bernoulli beam on a two-parameter elastic foundation

$\alpha_1 l^4$	$\alpha_2 l^2$											
	0	0	0	$0.5\pi^2$	$0.5\pi^2$	$0.5\pi^2$	π^2	π^2	π^2	$2.5\pi^2$	$2.5\pi^2$	$2.5\pi^2$
	Present method	[9, 11] [52] [53]	Naidu & Rao (FEM) [54]	Present method	[9, 11] [52] [53]	Naidu & Rao (FEM) [54]	Present method	[9, 11] [52] [53]	Naidu & Rao (FEM) [54]	Present method	[9, 11] 52 53	Naidu & Rao (FEM) [54]
0	9.8824	9.8696	9.8696	14.8172	14.804	14.804	19.752	19.739	19.739	34.5564	35.544	35.544
1	9.9837	9.9709	9.9709	14.9185	14.907	14.907	19.8533	19.841	19.841	34.677	35.645	35.645
100	20.0131	20.002	20.002	24.9479	24.937	24.937	29.8827	29.871	29.871	44.6871	44.676	44.676
1900	202.3657	201.41	201.41	207.3	206.35	206.35	212.235	211.28	211.28	227.04	226.0	226.0
10,000	1022.954	1023.1	–	1027.889	1028	–	1032.8230	1032.9	–	1047.628	1047.7	–

Table 2. Critical buckling load parameters of EBBo2PF clamped on both ends

$\alpha_1 l^4$	$\alpha_2 l^2$											
	0	0	0	$0.5\pi^2$	$0.5\pi^2$	$0.5\pi^2$	π^2	π^2	π^2	$2.5\pi^2$	$2.5\pi^2$	$2.5\pi^2$
	Present study	Rao & Raju [52]	FEM [51]	Present Study	Rao & Raju [55]	FEM [54]	Present study	Rao & Raju [55]	FEM [54]	Present study	Rao & Raju [55]	FEM [54]
0	42 (6.39 %)	39.478	39.479	46.9348 (5.68 %)	44.413	44.414	51.8696 (5.11 %)	49.348	49.349	66.674 (3.93 %)	64.152	64.153
1	42.0833 (6.39 %)	39.554	39.555	47.0181 (5.68 %)	44.489	44.490	51.9529 (5.12 %)	49.424	49.425	66.7573 (3.93 %)	64.228	64.229
100	50.3333 (6.92 %)	47.077	47.077	55.2681 (6.26 %)	52.012	51.542	60.2029 (5.72 %)	56.9471	56.877	75.0073 (4.54 %)	71.751	71.681

The values in parenthesis in the present study results are percentage differences between present results and previous results by Rao and Raju [55].

Table 3. Buckling load parameters of EBBo2PF with one end clamped and the other end simply supported

$\alpha_1 l^4$	$\alpha_2 l^2$					
	0	0	0	$0.5\pi^2$	π^2	$2.5\pi^2$
	Present method	Exact [36]	Atay and Coskun [43]	Present method	Present method	Present method
0	21 (4 %)	20.1907	20.1908	25.9348	30.8696	45.674
50	25.39815 (4.58 %)	24.2852	24.2855	30.3330	35.2678	50.0722
100	29.7963 (5.26 %)	28.3066	28.3080	34.7311	39.6659	54.4703

The figures in parenthesis under the present method results are percentage differences between present results and the exact results by Wang et al. [36].

6. Discussion

In this work, Galerkin variational method has been used for the critical buckling load determination of EBBo2PF. The governing equation is a homogeneous ordinary differential equation in terms of the buckled deflection $w(x)$. The paper presented the GVF in general form in terms of n parameter shape functions which satisfy the boundary conditions.

The Galerkin method was illustrated using one-parameter shape function formulation for three cases of boundary conditions, namely:

– Both ends are simply supported,

– Both ends are clamped ends and one-end is clamped and the other end is simply supported.

In each case, the GVF reduced to an algebraic equation which is solved to find the critical buckling load in standard form in terms of buckling load parameters $F_1(\alpha_1 l^4, \alpha_2 l^2)$, $F_2(\alpha_1 l^4, \alpha_2 l^2)$ and $F_3(\alpha_1 l^4, \alpha_3 l^2)$.

The buckling load parameters for simply supported EBBo2PF presented in Table 1 shows that the present results are almost identical to the exact results previously obtained by Ike [9, 11] and Naidu and Rao [54]. Table 2 presents critical buckling load parameters for EBBo2PF with both ends clamped, and shows that the present results differ from the results by Rao and Raju [55] by 3.93 % for $\alpha_2 l^2 = 2.5\pi^2$, $\alpha_1 l^4 = 0$ to 4.54 % for $\alpha_2 l^2 = 2.5\pi^2$, $\alpha_1 l^4 = 100$. Comparable differences exist for other values of $\alpha_1 l^4$ and $\alpha_2 l^2$ presented.

Table 3 compares the present results for $\alpha_2 l^2 = 0$, $\alpha_1 l^4 = 0$, 50, 100 with the exact results. Table 3 shows that the present results differ from the exact results by 4 % for $\alpha_1 l^4 = 0$, $\alpha_2 l^2 = 0$, and 5.26 % for $\alpha_1 l^4 = 100$, $\alpha_2 l^2 = 0$. The accuracy of the one-parameter Galerkin method is thus illustrated in the paper.

7. Conclusions

This paper has presented Galerkin method for solving critical buckling problems of EBBo2PF for the three cases of simply supported (SS) edges, clamped clamped (CC) edges and clamped-simply supported (CS) edges.

In conclusion:

1) The GVF simplified the problem to an algebraic eigenvalue problem.

2) One-parameter shape function formulation of the Galerkin solution for simply supported EBBo2PFs gave critical buckling load solutions that are almost identical to exact solutions.

3) One-parameter Galerkin solutions for clamped clamped EBBo2PF gave critical buckling loads that differ from the previous results by 3.93 % for $\alpha_2 l^2 = 2.5\pi^2$, $\alpha_1 l^4 = 0$ to 6.92 % for $\alpha_2 l^2 = 0$, $\alpha_1 l^4 = 100$.

4) The present one-parameter Galerkin solution for EBBo2PF with CS supports gave differences varying from 4 % for $\alpha_1 l^4 = 0$, $\alpha_2 l^2 = 0$ to 5.26 % for $\alpha_1 l^4 = 100$, $\alpha_2 l^2 = 0$.

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Data availability

The datasets generated during and/or analyzed during the current study are available from the corresponding author on reasonable request.

Conflict of interest

The authors declare that they have no conflict of interest.

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