Generalised Poisson-new linear-exponential distribution

Binod Kumar Sah[1], Suresh Kumar Sahani[2]
[1]Department of Statistics, R. R. M. Campus, Tribhuvan University, Janakpurdham, Nepal
[2]Department of Science and Technology, Rajarshi Janak University, Janakpurdham, Nepal
[2]Corresponding author
E-mail: [1]*sah.binod01@gmail.com*, [2]*sureshkumarsahani35@gmail.com*, [2]*sureshsahani@rju.edu.np*

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Abstract. It is a compound discrete probability distribution which has two parameters and whose particular case is Poisson-New Linear-Exponential Distribution (PNLED) [1]. The statistical characteristics required for this distribution such as probability mass function (pmf), statistical moments, estimation of parameters and goodness of fit have been derived and explained nicely. To test theoretical reliability of this distribution, goodness of fit has been applied to some over-dispersed data which were used by other researchers and further, we found that this distribution looks more appropriate for statistical modelling than PNLED, generalised Poisson-Lindley distribution (GPLD) [1, 2] and Poisson-Lindley distribution (PLD) [3].

Keywords: distribution, Poisson-new linear-exponential distribution, probability distribution, moments, proposed distribution, over-dispersed, goodness of fit.

1. Introduction

A generalisation of any probability distribution is considered more appropriate when it is more reliable than its particular case as well as previously obtained generalised distribution with similar conditions in all aspects.

The proposed distribution is a compound distribution. For compounding process, at least two distributions are required. We have needed Generalised Poisson distribution (GPD) [4] and New Linear-exponential distribution (NLED) [5] to conduct compounding process. GPD [4] distribution is a discrete distribution whereas NLED [5] is a continuous distribution. GPD [4] has two parameters and in mixing process it plays role of an original distribution and its original parameter λ follows NLED [5]. The pmf of GPD [4] has been given by:

$$P_1\,(Z;\,\lambda,\theta)=\frac{\lambda(\lambda+\theta z)^{z-1}e^{-(\lambda+\theta z)}}{z!}, \tag{1}$$

where $Z=0,1,2,\ldots;\ \lambda>0;\ |\,\theta|<1$.

The probability density function (pdf) of NLED [5] was given by:

$$f_2(z\,;\,\beta)=\frac{\beta^2}{(1+\pi\beta\,)}(\pi+\beta)e^{-\beta y},\quad z>0,\quad \beta>0. \tag{2}$$

Hence, Generalised Poisson-New Linear-exponential distribution (GPNLED) has been obtained by compounding GPD [4] with NLED [5]. The study of this papers mentioned in the following references [1, 2], [4], [6] have very much contributed to increase quality of the paper.

In this paper, the following works have been done. Construction and derivation of:

1) Probability Mass Function (pmf) of GPNLED.
2) Statistical Moments of GPNLED.
3) Estimation of Parameters of GPNLED.
4) Goodness of Fit and applications of GPNLED.

We have been studied the following papers to improve quality of this paper which have been mentioned in the following references [2], [7-14].

2. Results

The results obtained for this paper have been placed under the following sub-headings.

2.1. Probability mass function of GPNLED

GPD has two parameters λ and θ. λ is an original parameter of GPD which follows NLED. θ is an additional parameter of GPD which is versatile in nature and hence, GPD. Here, $\lambda > 0$ and $|\theta| < 1$. NLED has a single parameter β. The Probability mass of function of GPNLED is obtained by mixing GPD with NLED. This is how the pmf of GPNLED have been extracted:

$$\begin{aligned}
P\,(Z;\beta,\theta) &= \left[\frac{\beta^2 e^{-\theta z}}{z!\,(1+\pi\beta)}\right]\int_0^\infty \left\{\sum_{i=1}^{z-1}\binom{z-1}{i}\left(\frac{\theta z}{1}\right)^i (\pi\lambda^{z-i}+\lambda^{z-i+1})e^{-\lambda(1+\beta)}\right\} d\lambda \\
&= \left[\frac{\beta^2 e^{-\theta z}}{z!\;(1+\pi\beta)}\right]\left\{\sum_{i=0}^{z-1}\frac{(z-1)!}{i!\,(z-i-1)!}\left(\frac{\theta z}{1}\right)^i\left(\frac{\pi\Gamma(z-i+1)}{(1+\beta)^{z-i+1}}+\frac{\pi\Gamma(z-i+2)}{(1+\beta)^{z-i+2}}\right)\right\} \\
&= \left[\frac{\beta^2 e^{-\theta z}}{z(1+\pi\beta)}\right]\left\{\sum_{i=0}^{z-1}\left(\frac{\theta^i z^i}{i!\,(z-i-1)!}\right)\left(\frac{\Gamma(z-i+1)}{(1+\beta)^{z-i+1}}\right)\left(\pi+\frac{(z-i+1)}{(1+\beta)}\right)\right\} \\
&= \left[\frac{\beta^2 e^{-\theta z}}{(1+\pi\beta)}\right]\left\{\sum_{i=0}^{z-1}\left(\frac{\theta^i z^{i-1}}{i!}\right)\left(\frac{(z-i)}{(1+\beta)^{z-i+2}}\right)(\pi(1+\beta)+(z-i+1))\right\} \\
&= \left[\frac{\beta^2 e^{-\theta z}}{(1+\pi\beta)}\right]\left[\frac{\{\pi(1+\beta)+(z+1)\}}{(1+\beta)^{z+2}}\right] \\
&+ \left[\frac{\beta^2 e^{-\theta z}}{(1+\pi\beta)}\right]\left\{\sum_{i=1}^{z-1}\left(\frac{\theta^i z(z-i)^{i-1}}{i!}\right)\left(\frac{\{\pi(1+\beta)+(z-i+1)\}}{(1+\beta)^{z-i+2}}\right)\right\}
\end{aligned} \tag{3}$$

$z = 0,1,2,\ldots,\quad \lambda > 0,\quad \beta > 0.$

Probability mass function of GPNLED can be obtained for each value such as $z = 0,1,2,\ldots$ can be obtained by using the Eq. (3) as follows:

$$P(z=0) = \left[\frac{\beta^2}{(1+\pi\beta)}\right]\left[\frac{\{\pi(1+\beta)+1\}}{(1+\beta)^2}\right], \tag{4}$$

$$P(z=1) = \left[\frac{\beta^2 e^{-\theta}}{(1+\pi\beta)}\right]\left[\frac{\{\pi(1+\beta)+2\}}{(1+\beta)^3}\right], \tag{5}$$

$$P(z=2) = \left[\frac{\beta^2 e^{-2\theta}}{(1+\pi\beta)}\right]\left[\frac{\{\pi(1+\beta)+3\}}{(1+\beta)^4}+\frac{\theta\{\pi(1+\beta)+2\}}{(1+\beta)^3}\right], \tag{6}$$

$$P(z=3) = \left[\frac{\beta^2 e^{-3\theta}}{(1+\pi\beta)}\right]\left[\frac{\{\pi(1+\beta)+4\}}{(1+\beta)^5}+\frac{2\theta\{\pi(1+\beta)+3\}}{(1+\beta)^4}+\frac{1.5\theta^2\{\pi(1+\beta)+2\}}{(1+\beta)^3}\right], \tag{7}$$

$$\begin{aligned}
P(z=4) &= \left[\frac{\beta^2 e^{-4\theta}}{(1+\pi\beta)}\right]\left[\frac{\{\pi(1+\beta)+5\}}{(1+\beta)^6}+\frac{3\theta\{\pi(1+\beta)+4\}}{(1+\beta)^5}+\frac{4\theta^2\{\pi(1+\beta)+3\}}{(1+\beta)^4}\right. \\
&\left.+\frac{16\theta^3\{\pi(1+\beta)+2\}}{6(1+\beta)^3}\right],
\end{aligned} \tag{8}$$

$$P(z=5)=\left[\frac{\beta^2 e^{-5\theta}}{(1+\pi\beta)}\right]\left[\frac{\{\pi(1+\beta)+6\}}{(1+\beta)^7}+\frac{4\theta\{\pi(1+\beta)+5\}}{(1+\beta)^6}+\frac{15\theta^2\{\pi(1+\beta)+4\}}{2(1+\beta)^5}+\frac{50\theta^3\{\pi(1+\beta)+3\}}{6(1+\beta)^4}+\frac{125\theta^4\{\pi(1+\beta)+2\}}{24(1+\beta)^3}\right]. \tag{9}$$

2.2. Statistical moments of GPNLED

Let $\mu_r{}'$ denotes rth moment about origin of GPNLED:

$$\mu'_r=E[E(Z^r/\lambda]=\frac{\beta^2}{(1+\pi\beta)}\int_0^\infty\left(\sum_{z=0}^\infty\frac{z^r\lambda(\lambda+\theta z)^{z-1}e^{-(\lambda+\theta z)}\lambda^z}{\Gamma(z+1)}\right)(\pi+\lambda)e^{-\beta\lambda}d\lambda, \tag{10}$$

$$\begin{aligned}\mu'_1&=\frac{\beta^2}{(1+\pi\beta)}\int_0^\infty\left(\sum_{z=0}^\infty\frac{z^1\lambda(\lambda+\theta z)^{z-1}e^{-(\lambda+\theta z)}\lambda^z}{\Gamma(z+1)}\right)(\pi+\lambda)e^{-\beta\lambda}d\lambda\\&=\frac{\beta^2}{(1+\pi\beta)}\int_0^\infty\frac{\lambda}{(1+\theta)}(\pi+\lambda)e^{-\beta\lambda}d\lambda=\frac{(\pi\beta+2)}{\beta(1+\pi\beta)(1-\theta)}.\end{aligned} \tag{11}$$

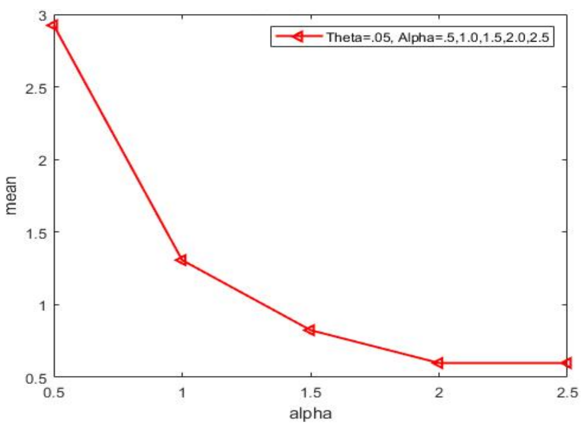

Fig. 1. Showing the mean of GPNLED

$$\begin{aligned}\mu'_2&=\frac{\beta^2}{(1+\pi\beta)}\int_0^\infty\left(\sum_{z=0}^\infty\frac{z^2\lambda(\lambda+\theta z)^{z-1}e^{-(\lambda+\theta z)}\lambda^z}{\Gamma(z+1)}\right)(\pi+\lambda)e^{-\beta\lambda}d\lambda\\&=\frac{\beta^2}{(1+\pi\beta)}\int_0^\infty\left\{\frac{\lambda}{(1-\theta)^3}+\frac{\lambda^2}{(1-\theta)^2}\right\}(\pi+\lambda)e^{-\beta\lambda}d\lambda\\&=\frac{[\beta(\pi\beta+2)+2(1-\theta)(\pi\beta+3)]}{\beta^2(\pi\beta+1)(1-\theta)^3}.\end{aligned} \tag{12}$$

$$\begin{aligned}\mu'_3&=\frac{\beta^2}{(1+\pi\beta)}\int_0^\infty\left(\sum_{z=0}^\infty\frac{z^3\lambda(\lambda+\theta z)^{z-1}e^{-(\lambda+\theta z)}\lambda^z}{\Gamma(z+1)}\right)(\pi+\lambda)e^{-\beta\lambda}d\lambda\\&=\frac{\beta^2}{(1+\pi\beta)}\int_0^\infty\left\{\frac{(1+2\theta)\lambda}{(1-\theta)^5}+\frac{3\lambda^2}{(1-\theta)^4}+\frac{\lambda^3}{(1-\theta)^3}\right\}(\pi+\lambda)e^{-\beta\lambda}d\lambda\\&=\frac{(1+2\theta)(\pi\beta+2)}{\beta(1+\pi\beta)(1-\theta)^5}+\frac{2(\pi\beta+3)}{\beta^2(1+\pi\beta)(1-\theta)^4}+\frac{6(\pi\beta+4)}{\beta^3(1+\pi\beta)(1-\theta)^5}.\end{aligned} \tag{13}$$

$$\begin{aligned}\mu'_4 &= \frac{\beta^2}{(1+\pi\beta)}\int_0^\infty \left(\sum_{z=0}^{\infty}\frac{z^4\lambda(\lambda+\theta z)^{z-1}e^{-(\lambda+\theta z)}\lambda^z}{\Gamma(z+1)}\right)(\pi+\lambda)e^{-\beta\lambda}d\lambda \\ &= \frac{\beta^2}{(1+\pi\beta)}\int_0^\infty \left\{\frac{(1+8\theta+6\theta^2)\lambda}{(1-\theta)^7}+\frac{(7+8\theta)\lambda^2}{(1-\theta)^6}+\frac{6\lambda^3}{(1-\theta)^5}+\frac{\lambda^4}{(1-\theta)^4}\right\}(\pi \\ &+\lambda)e^{-\beta\lambda}d\lambda \\ &= \frac{(1+8\theta+6\theta^2)(\pi\beta+2)}{\beta(1+\pi\beta)(1-\theta)^7}+\frac{2(7+8\theta)(\pi\beta+3)}{\beta^2(1+\pi\beta)(1-\theta)^6}+\frac{(6)(6)(\pi\beta+4)}{\beta^3(1+\pi\beta)(1-\theta)^5} \\ &+\frac{24(\pi\beta+5)}{\beta^4(1+\pi\beta)(1-\theta)^4}.\end{aligned} \tag{14}$$

Central moments of GPNLED:

$$\begin{aligned}\mu_2 = E(z^2)-\{E(z)\}^2 &= \frac{[\beta(\pi\beta+2)+2(1-\theta)(\pi\beta+3)]}{\beta^2(\pi\beta+1)(1-\theta)^3}-\left(\frac{(\pi\beta+2)}{\alpha(\pi\beta+1)(1-\theta)}\right)^2 \\ &= \frac{[\pi^2\beta^3+\pi^2\beta^2+3\pi\beta^2+(4\pi+2)\beta+2]-\theta(\pi^2\beta^3+4\pi\beta+2)}{[\beta(\pi\beta+1)]^2(1-\theta)^3}.\end{aligned} \tag{15}$$

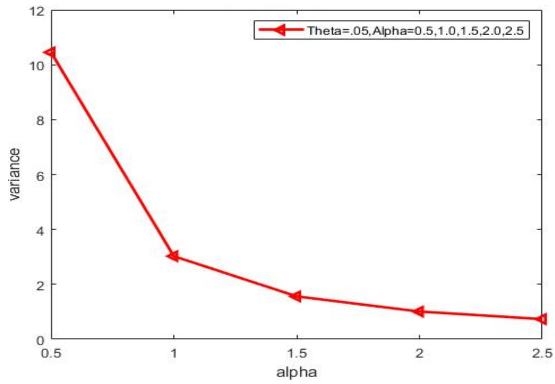

Fig. 2. Showing the variance of GPNLED

Proof. Variance > Mean.
Or:

$$\frac{[\pi^2\beta^3+\pi^2\beta^2+3\pi\beta^2+(4\pi+2)\beta+2]-\theta(\pi^2\beta^3+4\pi\beta+2)}{[\beta(\pi\beta+1)]^2(1-\theta)^3} > \frac{(\pi\beta+2)}{\beta(1+\pi\beta)(1-\theta)}.$$

Or:

$$(\pi^2\beta^3+3\pi\beta^2+2\beta)+(1-\theta)\{(\pi^2\alpha^2+4\pi\alpha+2)-(\pi\beta^2+2\beta)(1+\pi\beta)(1-\theta)\} > 0.$$

Or:

$$(1-\theta)\{(\pi^2\alpha^2+4\pi\alpha+2)-(\pi\beta^2+2\beta)(1+\pi\beta)(1-\theta)\} > 0.$$

Or:

$$(1-\theta) \le \frac{(\pi^2\beta^2+4\pi\beta+2)}{(\pi^2\beta^3+3\pi\beta^2+2\beta)}. \tag{16}$$

Hence the statement.

The third moment about the mean of this distribution can be obtained as:

$$\begin{aligned}
\mu_3 &= E(z^3) - 3E(z^2)E(z) + 2\{E(z)\}^3 \\
&= \left[\frac{(1+2\theta)(\pi\beta+2)}{\beta(1+\pi\beta)(1-\theta)^5} + \frac{2(\pi\beta+3)}{\beta^2(1+\pi\beta)(1-\theta)^4} + \frac{6(\pi\beta+4)}{\beta^3(1+\pi\beta)(1-\theta)^5}\right] \\
&\quad - 3\left[\frac{[\beta(\pi\beta+2)+2(1-\theta)(\pi\beta+3)]}{\beta^2(\pi\beta+1)(1-\theta)^3}\right]\left[\frac{(\pi\beta+2)}{\beta(1+\pi\beta)(1-\theta)}\right] \\
&\quad + 2\left[\frac{(\pi\beta+2)}{\beta(1+\pi\beta)(1-\theta)}\right]^3 \\
&= \frac{\begin{bmatrix} [\{\beta^2(1+\pi\beta)^2(1+2\theta)(\pi\beta+2)\} + \{3(1-\theta)\beta(1+\pi\beta)^2(2\pi\beta+6)\} \\ +\{(1-\theta)^2(1+\pi\beta)^2(6\pi\beta+24)\}] \\ -[\{3(1-\theta)\beta(1+\pi\beta)(\pi\beta+2)^2\}\{6(1-\theta)^2(1+\pi\beta)(\pi\beta+2)(\pi\beta+3)\} \\ +[\{2(\pi\beta+2)^3\}] \end{bmatrix}}{[\beta(1+\pi\beta)]^3(1-\theta)^5}.
\end{aligned} \tag{17}$$

The third central moment (μ_3) gives positive value, this distribution is positively skewed in shape.

$$\begin{aligned}
\mu_4 &= E(z^4) - 4E(z^3)E(z) + 6E(z^2)\{E(z)\}^2 - 3\{E(z)\}^4 \\
&= \frac{(1+8\theta+6\theta^2)(\pi\beta+2)}{\beta(1+\pi\beta)(1-\theta)^7} + \frac{2(7+8\theta)(\pi\beta+3)}{\beta^2(1+\pi\beta)(1-\theta)^6} + \frac{(6)(6)(\pi\beta+4)}{\beta^3(1+\pi\beta)(1-\theta)^5} \\
&\quad + \frac{24(\pi\beta+5)}{\beta^4(1+\pi\beta)(1-\theta)^4} - 4\left[\frac{(1+2\theta)(\pi\beta+2)}{\beta(1+\pi\beta)(1-\theta)^5} + \frac{2(\pi\beta+3)}{\beta^2(1+\pi\beta)(1-\theta)^4}\right. \\
&\quad \left. + \frac{6(\pi\beta+4)}{\beta^3(1+\pi\beta)(1-\theta)^5}\right]\left[\frac{(\pi\beta+2)}{\beta(1+\pi\beta)(1-\theta)}\right] \\
&\quad + 6\left[\frac{[\beta(\pi\beta+2)+2(1-\theta)(\pi\beta+3)]}{\beta^2(\pi\beta+1)(1-\theta)^3}\right]\left[\frac{(\pi\beta+2)}{\beta(1+\pi\beta)(1-\theta)}\right]^2 \\
&\quad - 3\left[\frac{(\pi\beta+2)}{\beta(1+\pi\beta)(1-\theta)}\right]^4 \\
&= \frac{\begin{bmatrix} [\{(1+8\theta+6\theta^2)\beta^3(1+\pi\beta)^3(\pi\beta+2)\} \\ +\{(7+8\theta)(1-\theta)\beta^2(1+\pi\beta)^3(2\pi\beta+6)\} \\ +\{6(1-\theta)^2\beta(1+\pi\beta)^3(6\pi\beta+24)\} + \{(1-\theta)^3(1+\pi\beta)^3(24\pi\beta+120)\}] \\ -4(1-\theta)(\pi\beta+2)[\{(1+2\theta)\beta^2(2+\pi\beta)\} \\ +\{3(1-\theta)\beta(2\pi\beta+6)\} + (1-\theta)^2(6\pi\beta+24)] \\ +6(1-\theta)^2(\pi\beta+1)(\pi\beta+2)^2[\{\beta(2+\pi\beta)+2(1-\theta)(3+\pi\beta)\}] \\ -3(1-\theta)^3(\pi\beta+2)^4 \end{bmatrix}}{[\beta(1+\pi\beta)]^4(1-\theta)^7}.
\end{aligned} \tag{18}$$

The Eq. (18) is obtained to know about nature of distribution according to size and it is the fourth central moment. Co-efficient of skewness and kurtosis based on moments can be obtained as follows.

$$\gamma_1 = \frac{\mu_3}{(\mu_2)^{3/2}} = \frac{\begin{bmatrix} [\{\beta^2(1+\pi\beta)^2(1+2\theta)(\pi\beta+2)\} + \{3(1-\theta)\beta(1+\pi\beta)^2(2\pi\beta+6)\} \\ +\{(1-\theta)^2(1+\pi\beta)^2(6\pi\beta+24)\}] \\ -[\{3(1-\theta)\beta(1+\pi\beta)(\pi\beta+2)^2\} + \{6(1-\theta)^2(1+\pi\beta)(\pi\beta+2)(\pi\beta+3)\}] \\ +[\{2(1-\theta)^2(\pi\beta+2)^3\}] \end{bmatrix}}{(1-\theta)^{1/2}[[\pi^2\beta^3+\pi^2\beta^2+3\pi\beta^2+(4\pi+2)\beta+2]-\theta(\pi^2\beta^3+4\pi\beta+2)]^{3/2}}. \tag{19}$$

From Eq. (19), it has been found that range of γ_1 is $(\sqrt{2}) < \gamma_1 < \infty)$. Hence, it is positively skewed:

$$\beta_2 = \frac{\mu_4}{(\mu_2)^2} = \frac{\begin{bmatrix} [\{(1+8\theta+6\theta^2)\beta^3(1+\pi\beta)^3(\pi\beta+2)\} \\ +\{(7+8\theta)(1-\theta)\beta^2(1+\pi\beta)^3(2\pi\beta+6)\} \\ +\{6(1-\theta)^2\beta(1+\pi\beta)^3(6\pi\beta+24)\} + \{(1-\theta)^3(1+\pi\beta)^3(24\pi\beta+120)\}] \\ -4(1-\theta)(\pi\beta+2)[\{(1+2\theta)\beta^2(2+\pi\beta)\} + \{3(1-\theta)\beta(2\pi\beta+6)\} \\ +(1-\theta)^2(6\pi\beta+24) \\ +6(1-\theta)^2(\pi\beta+1)(\pi\beta+2)^2[\{\beta(2+\pi\beta)+2(1-\theta)(3+\pi\beta)\}] \\ -3(1-\theta)^3(\pi\beta+2)^4 \end{bmatrix}}{(1-\theta)^{1/2}[[\pi^2\beta^3+\pi^2\beta^2+3\pi\beta^2+(4\pi+2)\beta+2]-\theta(\pi^2\beta^3+4\pi\beta+2)]^{3/2}}. \tag{20}$$

2.3. Estimation of parameters

This distribution has two parameters β and θ which can be obtained as
By using $P(z=0)$ and μ_1'. We have:

$$P(z=0) = \left[\frac{\beta^2}{(1+\pi\beta)}\right]\left[\frac{\{\pi(1+\beta)+1\}}{(1+\beta)^2}\right].$$

Solving it, we get:

$$f(\beta) = \pi(1-k)\beta^3 + (1+\pi-k-2\pi k)\beta^2 - k(2+\pi)\beta - k = 0. \tag{21}$$

The Polynomial Eq. (21) can be solved by using Regula-Falshi method or Newton Rapson method, where k denote $P(z=0)$.Substituting the estimated value of β in the following Eq. (22), we get an estimated value of θ:

$$(1-\theta) = \frac{(\pi\beta+2)}{[\mu_1'\beta(1+\pi\beta)]}. \tag{22}$$

μ_1' and μ_2': substituting the value of $(1-\theta)$ obtained from Eq. (22) in the expression of μ_2', we get Polynomial Eq. (23) in β which can be solve by Newton-Rapson or Regla-Falsi method:

$$f(\beta) = \mu_2'(\pi\beta+2)^3 - (\mu_1')^2(\pi\beta+1)(\pi\beta+2)\{\mu_1'\beta^2(\pi\beta+1) + 2(\pi\beta+3)\} = 0. \tag{23}$$

2.4. Goodness of fit and applications of GPNLED

This distribution can be used to test goodness of fit and to get possible inferential Statistics in the field related to accident proneness, risk management of production engineering, ecological sciences, agricultural fields related to events about insects, error per page, biological sciences, and other fields [15-17].

Applications of GPNLE model in the field of engineering. This distribution is applicable to fit better where Poisson, Generalised Poisson and Poisson mixtures of continuous distributions are applicable such as:

1) The count of α-particles emitted per unit of time is useful in analysis of any radio-active substances.

2) Number of telephone calls received on a given switch board per small unit of time.

3) In industrial production to find the proportion of defects per unit length, per unit area, etc.

4) In the field of reliability engineering.

In the following examples Chi-square goodness of fit have been applied by using GPNLE model.

Table 1. Example (1)

Z	0	1	2	3	4^+
Observed frequency	35	11	8	4	2

Table 2. Example (2)

Number of accidents	0	1	2	3	4	5^+
Observed frequency	447	132	42	21	3	2

Table 3. Example (3)

Z	0	1	2	3	4	5	6
Observed frequency	200	57	30	7	4	0	2

The first example is related to the number of errors per page which was given by Kemp and Kemp [18]. The second example is related to the data of accidents to 647 women working on H. E. Shells in 5 weeks was given by Greenwood and Yule [19]. The data related to Class per exposure (μg/kg) which is included in example (3) was given by Catcheside et al. [20].

The first and second examples have been used in the Doctoral Thesis. The theoretical frequencies obtained by using PLD, GPLD and GPNLED have been placed in Table 4 and 5 to make comparison. In Table 6, the theoretical frequencies obtained by using PLD, PNLED and GPNLED have been placed for comparison [1-3].

Table 4. Observed versus expected frequency of example (1)

Number of errors per page	Observed frequency	Expected frequency		
		PLD	GPLD	GPNLED
0	35	33.0	35.0	35.0
1	11	15.3	13.3	13.3
2	8	6.8	6.3	6.4
3	4	2.9	3.0	2.9
4^+	2	2.0	2.4	2.4
	60.0	60.0	60.0	647.0
μ_1'	0.7833333	–	–	–
μ_2'	1.85	–	–	–
$\hat{\beta}$	–	1.7434	1.9387733	1.642188744
$\hat{\theta}$	–	–	0.1174867	0.09640955

Table 5. Observed versus expected frequency of example (2)

Number of accidents	Observed frequency	Expected frequency		
		PLD	GPLD	GPNLED
0	447	441	447.0	447.0
1	132	143	1132.7	132.9
2	42	45	44.8	45.0
3	21	14	15.1	15.0
4	3	4	5.0	5.0
5^+	2	1	2.4	2.1
	647	647	647.0	647.0
μ_1'	0.4652241	–	–	–
μ_2'	0.9100646	–	–	–
$\hat{\beta}$	–	–	2.8563455	2.496217
$\hat{\theta}$	–	–	0.0523222	0.0415098

Table 6. Observed versus expected frequency of example (3)

Class per exposure (μg/kg)	Observed frequency	Expected frequency		
		PLD	PNLED	GPNLED
0	200	191.8	192.8	200.0
1	57	70.3	69.2	60.5
2	30	24.9	24.6	23.7
3	7	8.6	8.7	9.5
4	4	2.9	3.1	3.8
5	0	1.0	1.1	1.5
6	2	0.5	0.5	1.0
	300	300.0	300.0	300.0
μ_1'	0.55333333	–	–	–
μ_2'	1.2533333	–	–	–
$\hat{\beta}$	–	2.353339	2.0501155	2.256957115
$\hat{\theta}$	–	–	–	0.1002898421

3. Conclusions

1) This distribution will be over-dispersed if $(1-\theta) \leq \frac{(\pi^2\beta^2+4\pi\beta+2)}{(\pi^2\beta^3+3\pi\beta^2+2\beta)}$.

2) The range of γ_1 is $(\sqrt{2}) < \gamma_1 < \infty)$. So, it is positively skewed in shape.

3) The range of β_2 is $6 < \beta_2 < \infty$. So, it is leptokurtic by size.

4) From the Tables 4 and 5, it is found that P-value obtained by using GPNLED is greater than those obtained by using GPLD [2] and PLD [3]

5) From Table 6, what we have observed that the P-value obtained by using GPNLED is bigger than those of PNLED [1] and PLD [3]

6) Hence, it is suggested to apply GPNLED instead of PLD, GPLD and PNLED in similar situation in all aspects.

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Data availability

The datasets generated during and/or analyzed during the current study are available from the corresponding author on reasonable request.

Author contributions

Binod Kumar Sah: conceptualization, data curation, formal analysis, investigation, writing – original draft preparation. Suresh Kumar Sahani: conceptualization, data curation, formal analysis, investigation, supervision, writing – original draft preparation, writing – review and editing.

Conflict of interest

The authors declare that they have no conflict of interest.

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Dr. **Binod Kumar Sah** is, currently, working as Associate Professor in the Department of Statistics of Ramswarup Ramsager multiple campus, a constituent Campus of Tribhuvan University, Janakpur, Nepal. He has been associated with the teaching field of Statistics for the past 30 years. In 1995, he achieved the title of Master of Science [M.Sc.] in the distinction division, from the Central Department of Statistics, Kirtipur, Tribhuvan University, Nepal. In 2013, he received the title of Doctor of Philosophy (Ph.D) in Statistics from Patna University under supervision of Professor Dr. Amarendra Mishra. Dr. Sah, currently, has been working in fields of probability distributions and continuous mixtures of Poisson distribution as well as generalised Poisson distribution and has been written fifty research papers in national and international reputed journals and maximum of them are in Scopus.

Dr. **Suresh Kumar Sahani** is an Assistant Professor in the Department of Science and Technology at Rajarshi Janak University, Janakpurdham, Nepal. He has received his Ph.D. degree in 2017 in applied mathematics from Shri Venkateshwara University, U.P., India. His broad research area includes mathematical analysis, summability, approximation theory, and applied mathematics. He has published more than 150 research papers in International and National reputed journals. He is editor-in-chief, editors of many reputed international journals. He is bearing responsibility as an active member of RJU research Committee. He is co-guide of Ph.D. scholar. He has published many research books and textbooks. He has over 18 years of teaching experience. He is the recipient of Nepal Bidyabhushan ‘Ka’ award by former president of Nepal Bidyadevi Bhandari. He has delivered his papers in many international conferences.

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