Connected metric dimension of the class of ladder graphs

M. Iqbal Batiha[1], Mohamed Amin[2], Basma Mohamed[3], H. Iqbal Jebril[4]
[1, 4]Department of Mathematics, Al Zaytoonah University of Jordan, Amman 11733, Jordan
[1]Nonlinear Dynamics Research Center (NDRC), Ajman University, Ajman 346, United Arab Emirates
[2, 3]Mathematics and Computer Science Department, Faculty of Science, Menoufia University, Shebin El-Koom, Egypt
[1]Corresponding author
E-mail: [1]*iqbalbatiha22@yahoo.com*, [2]*achieveyourdream1990@gmail.com*, [3]*bosbos25jan@yahoo.com*, [4]*i.jebril@zuj.edu.jo*

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Abstract. Numerous applications, like robot navigation, network verification and discovery, geographical routing protocols, and combinatorial optimization, make use of the metric dimension and connected metric dimension of graphs. In this work, the connected metric dimension types of ladder graphs, namely, ladder, circular, open, and triangular ladder graphs, as well as open diagonal and slanting ladder graphs, are studied.

Keywords: connected metric dimension, ladder graphs, connected resolving set.

1. Introduction

In [1, 2], a connected resolving set of graphs was introduced recently. The shortest path between any two vertices $u, v \in V(G)$ in a linked graph $G = (V, E)$ is represented by $d(u, v)$. An ordered vertex set $B = \{x_1, x_2, \dots, x_k\} \subseteq V(G)$ is a metric basis of G if B has minimum cardinality and the following representation:

$$r(v|\, B) = (d(v, x_1), d(v, x_2), \dots, d(v, x_k)), \tag{1}$$

is unique for each $v \in V(G)$. A metric basis B of G is connected if the subgraph $\overline{B}$ produced by B is a nontrivial connected subgraph of G. The metric dimension and connected metric dimension of G, denoted as $dim(G)$ and $cdim(G)$, respectively, have the following definitions: Let $|B|$ be the cardinality of B, then we have: $dim(G) = min\,\{|Bi|: Bi\ \subseteq 2^v,\ Bi$ is a resolving set of $G\}$, $cdim(G) = min\,\{|Bi|: Bi\ \subseteq 2^v,\ Bi$ is a connected resolving set of $G\}$.

The connected metric dimension at a vertex $v \in V(G)$, denoted as $cdim_G(v)$, is the metric basis of G that contains v and generates a connected subgraph of G; then:

$$cdim(G) = min_{v \in V\,(\mathrm{G})}\{cdim_G(v)\}. \tag{2}$$

Slater [3, 4] introduced the concept of metric basis as a locating set of G and uses the cardinality of B as a locating number to locate an intruder in a network. Harary and Melter independently discovered the ideas of a metric basis as a minimum resolving set of G and the cardinality of B as a metric dimension [5]. A number of graphs' metric dimensions are found in [6–15]. The following are the graphs: corona product [6], regular bipartite [7], chain graphs [8], mobius ladder [9], circulant graphs and Cayley hypergraphs [10], heptagonal circular ladder [11], generalized Petersen multigraphs [12], power of total graph [13], friendship graph [14], and quartz graph [15]. Specifically, the authors in [1] studied the linked metric dimension of the $n-1$ star network K_1, full graph K_n, cycle graph C_n, route graph P_n, and wheel graph W_n. The findings show that the cycle graph C_n, $n \geq 3$, and the route graph P_n, $n \geq 2$, both have connected metric dimensions that are equal to 2, whereas it is for the star graph $K_{1,n-1}$, $n \geq 4$, is $n-1$, and for the complete graph K_n, $n \geq 3$, is $n-1$. According to [2], the connected metric dimension of the wheel graph W_n, $n \geq 7$, is $\left\lfloor \frac{2n+2}{5} \right\rfloor + 1$, while it is for the Petersen graph P is 4, and if v is an end vertex of the tree T, then it is at a vertex of T; otherwise, it is at a vertex of T that is 2. Further information might be found in the literature [16-22], and some future notions may be applied to some applications like [23, 24].

To illustrate the ideas of metric dimension with its connected one, we plot in Fig. 1 the $3C_4$ – snake graph.

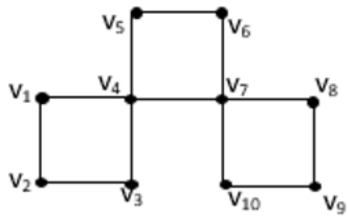

Fig. 1. $3C_4$-Snake graph G with $dim(G) = 3$ and $cdim(G) = 5$

The set $B = \{ v_1, v_5, v_8\}$ is a metric basis of G, due to the representations for the vertices of G are distinct. Since the representations $r(v_1|B) = (0,2,3)$, $r(v_2|B) = (1,3,4)$, $r(v_3|B) = (2,2,3)$, $r(v_4|B) = (1,1,2)$, $r(v_5|B) = (2,0,3)$, $r(v_6|B) = (3,1,2)$, $r(v_7|B) = (2,2,1)$, $r(v_8|B) = (3,3,0)$, $r(v_9|B) = (4,4,1)$, $r(v_{10}|B) = (3,3,2)$ for the vertices of G are different, the set $B = \{v_1, v_5, v_8\}$ is a metric basis of G. Therefore, $dim(G) = 3$, and the subgraph induced by $\bar{B} = (B, E)$ is disconnected. As a result, B and G are not connected resolving sets. To be more precise, no 3-element subset of G is a connected resolving set. Given that the representations $r(v_1|\bar{B}) = (0,1,2,2,3)$, $r(v_2|\bar{B}) = (1,2,3,3,4)$, $r(v_3|\bar{B}) = (2,1,2,2,3)$, $r(v_4|\bar{B}) = (1,0,1,1,2)$, $r(v_5|\bar{B}) = (2,1,0,2,3)$, $r(v_6|\bar{B}) = (3,2,1,1,2)$, $r(v_7|\bar{B}) = (2,1,2,0,1)$, $r(v_8|\bar{B}) = (3,2,3,1,0)$, $r(v_9|\bar{B}) = (4,3,4,2,1)$, $r(v_{10}|\bar{B}) = (3,2,3,1,2)$ are distinct, the set $\bar{B} = \{v_1, v_4, v_5, v_7, v_8\}$ is a connected resolving set. Therefore, $cdim(G) = 5$. However, we aim in this article to examine the connected metric dimension of a class of ladder graphs that includes slanting, open diagonal, triangular, open, circular, and open ladder graphs.

2. Applications of ladder graph

We present three applications pertaining to ladder graphs in this section. The three uses listed above and many more inspire us to study the uniquely elegant labelling for the ladder graph. The first application that is filed is in electronics. Resistor ladder networks are a quick and inexpensive way to do digital-to-analog conversion (DAC). The binary weighted ladder and the R/2R ladder are the two most widely used networks. Both devices can convert digital voltage information to analogue, but due to its greater accuracy and simple construction, the R/2R ladder has gained more popularity. The second application is electrical technology. The ladder flow graph is made using Ohm's equation and the two Kirchhoff equations. After that, the graph is inverted so that only forward paths are visible. The reciprocal of the total number of paths that could possibly lead from the output to the input node is the transfer function in the case of a simple ladder. If ladders with internal generators are dependent or independent, the transfer function can be found using similar methods with slight adjustments. Other relations, such as the input impedance and transfer admittance, can also be found using the flow graph. The third application is wireless communication. Over time, an increasing number of wireless networks have been developed to provide wire-free communication between any two devices (computers, phones, etc.). Nevertheless, there aren't enough radio frequencies accessible for wireless communication (just 11 channels in the 2.4 GHz range are available for all WiFi transmissions in the US). It is essential to provide a workable manner to offer safe communications in industries like phones, mobile,

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security systems, WiFi, and many others [25]. It annoys me when someone else calls while I'm on the phone. This irritation is caused by interference from uncontrolled simultaneous transmissions [26]. Resonance or interference between two sufficiently adjacent channels can cause damage to communication. Interference can be prevented with the proper channel assignment. To arrive at a solution, Hale [27-30] conceptualized the problem as a graph (vertex) coloring model, which we now refer to as the $L(2,1)$ coloring. Consider about the different transmitters and stations that are out there. In order to minimize interference, a channel must be assigned to each transmitter or station. Time-sensitive or error-sensitive communication may suffer from the interference phenomenon if transmitters are placed too close to one another either physically or through the frequency channels they use. To minimize interference, any two "close" transmitters ought to differ as much as possible from one another.

3. Results

Note that the two paths P_2 and P_n are the Cartesian products of a ladder graph L_n.

Theorem 3.1. We have $cdim(L_n) = \frac{n}{2}$, $n \geq 4$.

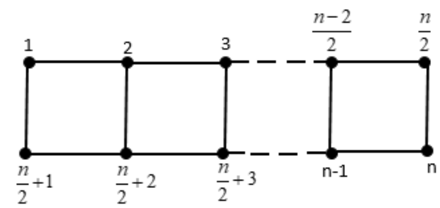

Fig. 2. Ladder graph L_n

Proof. Herein, we select $\bar{B}$ as $\bar{B} = \left\{v_1, v_2, \dots, v_{\frac{n-2}{2}}, v_{\frac{n}{2}}\right\}$. In this regard, we shall study each of the vertex representations of $v_i \in V(L_n)$ such that $n = 2k + 2$ for $n \geq 4$ with respect to $\bar{B}$. In other words, we have:

$$r(v_i|\bar{B}) = \begin{cases} (0,1,2,\dots,,k), & i = 1, \\ (i-1, i-2, \dots, 0, 1, \dots, k-i+1), & 2 \leq i \leq k, \\ (i-1, i-2, \dots, 1, 0), & i = k+1, \\ (1, 2, 3, \dots, k+1), & i = k+2, \\ (i-k-1, i-k-2, \dots, 1, 2, \dots, n-i+1), & k+3 \leq i \leq n-1, \\ (k+1, k, k-1, \dots, 2, 1), & i = n. \end{cases} \quad (3)$$

The set $\bar{B} = \left\{v_1, v_2, .., v_{\frac{n-2}{2}}, v_{\frac{n}{2}}\right\}$ has an induced subgraph that is connected, and as previously demonstrated, the vertices in graph L_n have unique representations. Despite not always being the lowest limit, this implies that $\bar{B}$ is a connected resolving set. Therefore, $cdim(L_n) \leq \frac{n}{2}$ is an upper bound. Thus, $cdim(L_n) \geq \frac{n}{2}$ is demonstrated. Given a connected resolving set $\bar{B} = \left\{v_1, v_2, \dots, v_{\frac{n-2}{2}}, v_{\frac{n}{2}}\right\}$, we assume that $|\bar{B}| = \frac{n}{2}$, and $\bar{B}_1$ is another minimum connected resolving set.

In the case that we choose an ordered set:

$$\bar{B}_1 \subseteq \bar{B} - \{v_i, v_j\}, \quad 1 \leq i, \quad j \leq \frac{n}{2}, \quad i \neq j, \quad (4)$$

for which the two vertices v_i, $v_j \in L_n$ are exist so that:

$$r(v_i|\bar{B}) = r(v_j|\bar{B}) = \left(\frac{n-2}{2}, \frac{n-2}{2} - 1, \frac{n-2}{2} - 2, \ldots, 1\right). \tag{5}$$

This is not the case; $\bar{B}_1$ is not a connected resolving set as assumed. The lowest bound is therefore $cdim\ (L_n) \geq \frac{n}{2}$.

Corollary 3.2. If the ladder graph ($C(L_n)$, $n \geq 4$, $n = 2k + 2$, is circular, then:

$$cdim(C(L_n)) \ = \begin{cases} \frac{n-k-1}{2}, & k \text{ is odd,} \\ \frac{n-k}{2}, & k \text{ is even.} \end{cases} \tag{6}$$

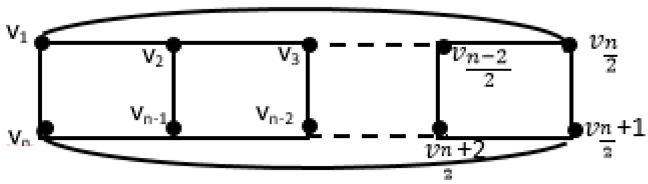

Fig. 3. Circular Ladder graph $C(L_n)$

Theorem 3.3. If the ladder graph $O(L_n)$, $n \geq 8$ is open, then $cdim\big(O(L_n)\big) = \frac{n}{2}$.

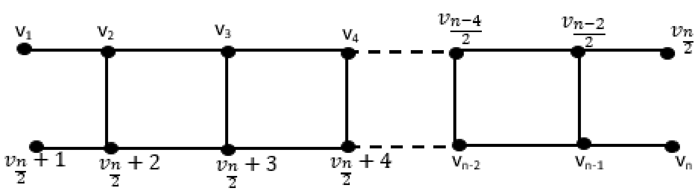

Fig. 4. Open ladder graph $O(L_n)$

Proof. Consider we have $\bar{B} = \left\{v_1, v_2, \ldots, v_{\frac{n-2}{2}}, v_{\frac{n}{2}}\right\}$. With respect to $\bar{B}$, we take into consideration the following representations of vertices $v_1 \in V\big(O\ (L_n)\big)$, $n \geq 8$, $n = 2k + 6$. So, we have:

$$r(v_i|\bar{B}) = \begin{cases} (0,1,2,\ldots,k+2), & i = 1, \\ (i-1, i-2, \ldots, 0, 1, \ldots, k-i+2, k-i+3)\,, & 2 \leq i \leq k+2, \\ (i-1, i-2, \ldots, 1, 0), & i = k+3, \\ (3, 2, 3, \ldots\ , k+3), & i = k+4, \\ (i-2k+1, i-2k, \ldots, 1, 2, \ldots, n-i+1), & k+5 \leq\ i \leq n-1, \\ (k+2, k+1, k, k-1, \ldots\ , 3, 2, 3), & i = n. \end{cases} \tag{7}$$

Although the induced subgraph of $\bar{B}$ is clearly connected and the representations of the vertices in graph $O(L_n)$ are different, this does not necessarily imply that $\bar{B}$ is a connected resolving set. For this reason, $cdim(O\ (L_n)) \leq \frac{n}{2}$ is an upper bound.

We now demonstrate that $cdim(O\ (L_n)) \leq \frac{n}{2}$. Given a connected resolving set $\bar{B} = \left\{v_1, v_2, \ldots, v_{\frac{n-2}{2}}, v_{\frac{n}{2}}\right\}$ with $|\bar{B}| = \frac{n}{2}$. Let $\bar{B}_1$ be an additional minimal connected resolving set. Let us choose an ordered set:

$$\bar{B}_1 \subseteq \bar{B} - \{v_i, v_j\}, \quad 1 \le i, \quad j \le \frac{n}{2}, \quad i \ne j, \tag{8}$$

for which there are two vertices v_i, $v_j \in O(L_n)$ such that:

$$r(v_i|\bar{B}) = r(v_j|\bar{B}) = \left(\frac{n}{2} - 1, \frac{n}{2} - 2, \frac{n}{2} - 3, \dots, 1\right). \tag{9}$$

It is not the case that $\bar{B}_1$ is a connected resolving set, as implied. Thus, $cdim(O\ (L_n)) \ge \frac{n}{2}$ is the lower bound. Ultimately, $cdim\big(O\ (L_n)\big) = \frac{n}{2}$.

Theorem 3.4. For $n \ge 4$, $cdim(TL_n) = \frac{n}{2}$ for which G is a triangular ladder graph TL_n.

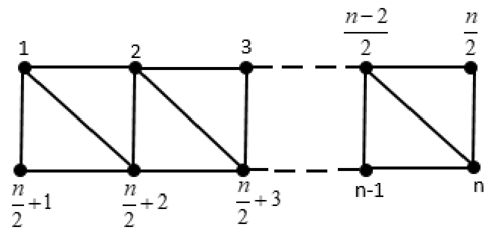

Fig. 5. Triangular Ladder graph TL_n

Proof. For $n \ge 4$ such that $n = 2k + 2$, we take into consideration a connected resolving set of (TL_n) as the set $\bar{B} = \left\{v_1, v_2, \dots, v_{\frac{n-2}{2}}, v_{\frac{n}{2}}\right\}$. The vertices' representations $v_i \in V(TL_n)$ in connection to $\bar{B}$ are given as follows:

$$r(v_i|\bar{B}) = \begin{cases} (0,1,2,\dots,k), & i = 1, \\ (i-1, i-2, \dots, 0, 1, \dots, k-i, k-i+1), & 2 \le i \le k, \\ (k, k-1, \dots, 1, 0), & i = k+1, \\ (1, 2, \dots, k, k+1), & i = k+2, \\ (i-n+k, i-n+k-1, \dots, 1, 1, 2, \dots, n-i+1), & k+3 \le i \le n. \end{cases} \tag{10}$$

The unique vertex representations in graph TL_n suggest that $\bar{B}$ is a connected resolving set, and as can be seen above, the induced subgraph of $\bar{B}$ is indeed connected, but this is not always the case. Therefore, $cdim(TL_n) \le \frac{n}{2}$ is an upper bound. Thus, we demonstrate that $cdim(TL_n) \ge \frac{n}{2}$.

Given a connected resolving set $\bar{B} = \left\{v_1, v_2, \dots, v_{\frac{n-2}{2}}, v_{\frac{n}{2}}\right\}$, $|\bar{B}| = \frac{n}{2}$. Let $\bar{B}_1$ be an additional minimal connected resolving set. When we choose an ordered set:

$$\bar{B}_1 \subseteq \bar{B} - \{v_i, v_j\}, \quad 1 \le i, \quad j \le \frac{n}{2}, \quad i \ne j. \tag{11}$$

Obviously, we can see that $\exists v_i$, $v_j \in TL_n$ for which:

$$r(v_i|\bar{B}) = r(v_j|\bar{B}) = \left(\frac{n-2}{2}, \frac{n-2}{2} - 1, \frac{n-2}{2} - 2, \dots, 1\right). \tag{12}$$

It is not the case that $\bar{B}_1$ is a connected resolving set, as implied. Hence, the lower bound is $cdim\ (TL_n) \ge \frac{n}{2}$. Therefore, we have $cdim\ (TL_n) = \frac{n}{2}$.

Corollary 3.5. For an open triangular ladder $O(TL_n)$, we have $cdim\big(O(TL_n)\big) = 3$, $n \geq 8$.

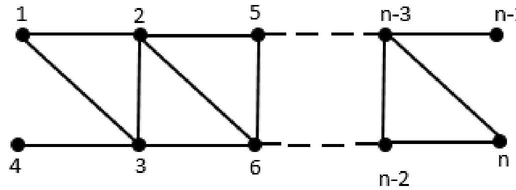

Fig. 6. Open triangular Ladder $O(TL_n)$

Theorem 3.6. If SL_n is a Slanting ladder graph, then $cdim(SL_n) = \frac{n}{2}$, for $n \geq 6$.

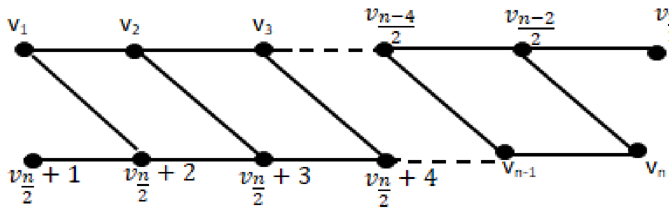

Fig. 7. Slanting Ladder SL_n

Proof. By taking into consideration $v_i \in V(SL_n)$ with respect to $\bar{B}$, we assume that $\bar{B} = \left\{v_1, v_2, \dots, v_{\frac{n-2}{2}}, v_{\frac{n}{2}}\right\}$ is a connected resolving set for (SL_n) in which $n \geq 6$, $n = 2k + 4$. This implies:

$$r(v_i|\bar{B}) = \begin{cases} (0,1,2,\dots,k+1), & i = 1, \\ (i-1, i-2, \dots, 0, 1, \dots, k-i+2), & 2 \leq i \leq k+2, \\ (2,3,\dots,k+3), & i = k+3, \\ (1,2,\dots,k+2), & i = k+4, \\ (i-2k+1, i-2k, \dots, 1, 2 \dots, n-i+2), & k+5 \leq i \leq n. \end{cases} \quad (13)$$

The unique vertex representations in graph SL_n suggest that $\bar{B}$ is a connected resolving set, and as can be shown above, the induced subgraph of $\bar{B}$ is clearly connected, but this is not always the case we desire. Because of this, $cdim(SL_n) \leq \frac{n}{2}$ is an upper bound. Thus, we demonstrate that $cdim(SL_n) \geq \frac{n}{2}$.

Given a connected resolving set $\bar{B} = \left\{v_1, v_2, .., v_{\frac{n-2}{2}}, v_{\frac{n}{2}}\right\}$, with $|\bar{B}| = \frac{n}{2}$. Let $\bar{B}_1$ be an additional minimal connected resolving set. In the event that we choose the following ordered set:

$$\bar{B}_1 \subseteq \bar{B} - \{v_i, v_j\}, \quad 1 \leq i, \quad j \leq \frac{n}{2}, \quad i \neq j, \quad (14)$$

for which $\exists v_i, v_j \in SL_n$ satisfying:

$$r(v_i|\bar{B}) = r(v_j|\bar{B}) = \left(\frac{n-2}{2}, \frac{n-2}{2} - 1, \frac{n-2}{2} - 2, \dots, 1\right). \quad (15)$$

It will not be a connected resolving set if, in contrast to the assumption, we choose an ordered $\bar{B}_1$. As a consequence, the lower bound is $cdim(SL_n) \geq \frac{n}{2}$, and hence $cdim(SL_n) = \frac{n}{2}$.

Theorem 3.7. If the graph $O(DL_n)$ is an open diagonal ladder, then $cdim(O(DL_n)) = \frac{n}{2}$,

$n \geq 6$.

Proof. Take into consideration the connected resolving set of $(O(DL_n))$ for which $n \geq 6$, $n = 2k + 6$, and the set $\bar{B} = \left\{v_1, v_2, \dots, v_{\frac{n-2}{2}}, v_{\frac{n}{2}}\right\}$. With respect to $\bar{B}$, consider $vi \in V(O(DL_n))$ as follows:

$$r(v_i|\bar{B}) = \begin{cases} (0,1,2,\dots,k+2), & i = 1, \\ (i-1, i-2, \dots, 0, 1, \dots, 2k-i-2, 2k-i-1), & 2 \leq i \leq k+2, \\ (i-1, i-2, \dots, 1, 0), & i = k+3, \\ (i-2, i-3, \dots, 1, 2), & i = k+4, \\ (n-i, n-i-1, \dots, 1, 1, 1 \dots, i-2k), & k+5 \leq i \leq n-1, \\ (2, 1, 2, 3, \dots, k+2), & i = n. \end{cases} \tag{16}$$

In graph $O(DL_n)$, the vertex representations are unique and the induced subgraph of $\bar{B}$ is undoubtedly connected, as can be observed above. This suggests that $\bar{B}$ is a connected resolving set, though it need not be the lowest bound. For this reason, $cdim(O(DL_n)) \leq \frac{n}{2}$ is an upper bound, and thus we demonstrate that $cdim(O(DL_n)) \geq \frac{n}{2}$.

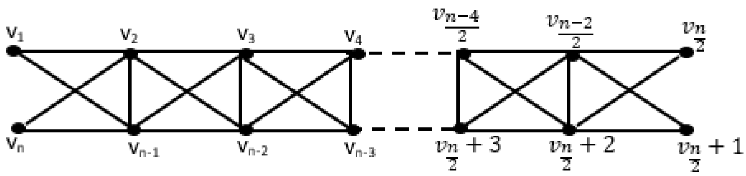

Fig. 8. Open diagonal ladder graph $O(DL_n)$

Given a connected resolving set $\bar{B} = \left\{v_1, v_2, \dots, v_{\frac{n-2}{2}}, v_{\frac{n}{2}}\right\}$ with $|\bar{B}| = \frac{n}{2}$. Let $\bar{B}_1$ be an extra minimal connected resolving set. Assume that we choose the following ordered set:

$$\bar{B}_1 \subseteq \bar{B} - \{v_i, v_j\}, \quad 1 \leq i, \quad j \leq \frac{n}{2}, \quad i \neq j, \tag{17}$$

for which $\exists v_i, v_j \in O(DL_n)$ satisfying:

$$r(v_i|\bar{B}) = r(v_j|\bar{B}) = \left(\frac{n-2}{2}, \frac{n-2}{2} - 1, \frac{n-2}{2} - 2, \dots, 1\right). \tag{18}$$

It will not be a connected resolving set if, in contrast to the assumption, we choose an ordered $\bar{B}_1$. As a consequence, the lower bound is $cdim(O(DL_n)) \geq \frac{n}{2}$, and hence $cdim(O(DL_n)) = \frac{n}{2}$.

4. Conclusions

The constant metric dimension of an open triangular ladder $O(TL_n)$ is 3. There are several types of ladder graphs with unbounded metric dimensions as $n \to \infty$, circular, open, triangular, slanting, and open diagonal. Our approach will be extended to ladder-class graph subdivisions in the future.

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Data availability

The datasets generated during and/or analyzed during the current study are available from the corresponding author on reasonable request.

Author contributions

M. Iqbal Batiha: conceptualization, validation, data curation. Mohamed Amin: methodology, formal analysis, supervision. Basma Mohamed: software, investigation. Iqbal H. Jebril: resources, visualization, supervision.

Conflict of interest

The authors declare that they have no conflict of interest.

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Iqbal M. Batiha holds a M.Sc. in Applied Mathematics (2014) from Al Al-Bayt University and a Ph.D. (2019) from The University of Jordan. He is a founding member of the International Center for Scientific Research and Studies (ICSRS, Jordan), and he is currently working as an assistant professor at the Department of Mathematics at Al-Zaytoonah University of Jordan, as well as at the Nonlinear Dynamics Research Center (NDRC) that was recently established at Ajman University. He has published several papers in different peer-reviewed international journals. Iqbal M. Batiha was awarded several prizes, including the Riemann-Liouville Award, which was presented at the International Conference on Fractional Differentiation and its Applications (ICFDA'18) that was held in Amman in July 2018, and the Oliviu Gherman Award, which was presented at the First Online Conference on Modern Fractional Calculus and Its Applications that was held in Turkey in December 2020.

Mohamed Amin received the B.Sc. degree in mathematics from the Faculty of Science, Menoufia University, in 1983, the M.Sc. degree in computer science from the Faculty of Science, Ain Shams University, in 1990, and the Ph.D. degree in computer science from Gdansk University, Poland, in1997. He is currently a Professor of computer science and the Head of the Department of Mathematics and Computer Science with Menoufia University. He has authored or coauthored many articles in international repute journals, such as Information Sciences, Scientific Reports-Nature, and others. His research interests include grammar systems as a link between AI and compiler design, metaheuristic optimization algorithms, Petri nets, reasoning dynamic fuzzy systems, cryptography, quantum information processing, image processing, and biometrics. He is a reviewer in many prestigious journals.

Basma Mohamed received the B.S. and M.S. degrees and Ph.D. degrees in computer science from the Faculty of Science, Menoufia University, in 2011, 2017, and 2023, respectively. In addition, she has over nine years of teaching and academic experience. Her research interests include graph theory, discrete mathematics, algorithm analysis, and meta-heuristic algorithms.

Iqbal H. Jebril is Professor at the Department of Mathematics, Al-Zaytoonah University of Jordan, Amman, Jordan. He obtained his Ph.D. in 2005 from the National University of Malaysia (UKM). His fields of interest include functional analysis, operator theory, and fuzzy logic. He had several prestigious journal and conference publications and was on various journals and conferences' committees.

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