A bulk queue's analysis with two-stage heterogeneous services, multiple vacations, closedown with server breakdown, and two types of renovation

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Abstract. A queueing system with two stages of heterogeneous services, multiple vacations, and closedown upon server failure is considered in this study for $M_X/G(a,b)$ data. Two different services must be provided to a group of customers one at a time. At time C, the server finishes its shut-down procedures following the conclusion of each SPS. A server that has a queue length shorter than a takes numerous, different-duration vacations. Upon returning from vacation, if the server detects that there are less than a customers in the queue, they will continue to serve the following batch if the server determines that there are at least a customers waiting for service. During the server's first phase of operation, the service channels will momentarily fail with a probability of (π_1) at any time, and during the server's second phase of operation, the service channels will fail with a probability of (π_2) at any time.

Keywords: bulk service, heterogeneous services, multiple vacations, closedown, queue size distribution, server breakdown.

1. Introduction

Queues with server vacations are a crucial component of queueing theory and have been thoroughly and fruitfully researched due to their various applications in manufacturing systems, communication systems, textile, food, and chemical processing industries, among other industries. In a traditional vacation queue, a server can entirely halt service or perform additional work while away. Offering several vacation rules gives the system's design and operation control additional flexibility.

B. T. Doshi [1] have discussed a queueing system with vacations (1986). H. S. Lee [2] Steady state probabilities for the server vacation model with group arrivals and under control operation policy (1991). H. Takagi [3] developed a foundation of performance evaluation with queueing analysis (1991). Madan K. C. [4] considered the M/G/1 queue with second optional service (2000). K. C. Madan [5] On a single server queue with two-stage heterogeneous service and deterministic server vacations (2001). Medhi [6] generalized the model by considering that the second optional service is also governed by a general distribution (2002).

Madan et al. [7] considered the classical M/G/1 queueing system in which the server provides the first essential service to all the arriving customers whereas some of them receive second optional service (2003). Arumuganathan and Jeyakumar [8] have studied Bulk queueing models with different parameters. The queueing model with two phases of heterogeneous service under Bernoulli schedule and a general vacation time is considered Madan and Choudhury [9]. After first-stage service the server must provide the second stage service (2005).

The various system performance measures for optimization of the T policy M/G/1 queue with server breakdowns and general startup times was presented by Wang et al. [10] (2007). An M/G/1 queue with two phases of service subject to the server breakdown and delayed repair has studied by Choudhury, Tadj and M. Paul [11]. This model generalized both the classical M/G/1 queue subject to random breakdown and delayed repair as well as an M/G/1 queue with second optional service and server breakdowns (2007).

Wang et al [12] have presented the various system performance measures for optimization of the Tpolicy M/G/1 queue with server breakdowns and general startup times (2007). Jain and Agrawal [13] analyzed the optimal policy for bulk queue with multiple types of server breakdown. In this paper breakdown occurs only when the server is in busy state and each type of breakdown requires a random number of finite stages of repair (2009).

Choudhury et al. [14] deals with an M/G/1 queueing system with two phases of service and Bernoulli vacation schedule for an unreliable server, which consist of a breakdown period and a delay period, under N –policy and a random setup time (2009). Jain, M. and Upadhyaya, S. [15] considered the Optimal repairable MX /G/1 queue with multi-optional services and Bernoulli vacation (2010).

Thangaraj and Vanitha [16] have analyzed a single server queue with Poisson arrivals, two stages of heterogeneous service with different service time distributions subject to random breakdowns and compulsory server vacations with general vacation periods (2010). The steady state behaviour of a batch arrival queue with two phases of heterogeneous service along and Bernoulli schedule vacation under multiple vacation policy is examined Choudhury et al. [17] (2011).

M. Balasubramanian and R.Arumuganathan [18] Steady state analysis of a bulk arrival general bulk service queueing system with modified M- vacation policy and variant arrival rate (2011). Jeyakumar and Senthilnathan [19] have discussed a study on the behaviour of the server breakdown without interruption in a $M_X/G(a\ ,b)/1$ queueing system with multiple vacations and closedown time (2012). Sourav Pradhan and Prasenjit Karan [20] Performance analysis of an infinite-buffer batch- size-dependent bulk service queue with server breakdown and multiple vacation (2022).

In the literature of the queueing system, few authors only have discussed about the repair or renovation due to breakdown of the service station. Practically, in many cases the renovation of the service station due to breakdown may be required. Such breakdowns have a specific effect on the system, particularly on the queue length, busy period of the server and waiting time of the customers.

For the first time, to our knowledge here the generally distributed variable batch size service and bulk arrival queueing system is analyzed in two - stage heterogeneous queueing system. It is important to note that in the literature of two - stage heterogeneous with bulk queueing models, only bulk arrival is considered. Paper on bulk service two - stage heterogeneous does not exist in the literature which is the motivation for the development of this paper. Our paper differs from the existing ones in the following way: Two-stage heterogeneous concept is newly considered for a variable batch size service queuing model which has more practical importance. Probability generating function of queue length distribution at an arbitrary time epoch in steady state is obtained by using supplementary variable technique by Lee's method.

The paper is structured as follows: 1. Introduction; 2. The queueing model description and its corresponding steady state equations are presented; 3. The distribution of queue sizes is covered; 4. PGF of the queue size at different epochs; 5. The queueing system's anticipated queue length is discovered; 6. Conclusion.

2. Model description and system equations

Crude oil refinery is one real-world use for the single server design. Petroleum contains a variety of chemical compounds. The process of purging impurities and dividing petroleum into products that may be used is called refining. There are two major phases in the refining process: phase I is where water is separated and sulfur compounds are removed, and phase II is where the fractionation process takes place.

Crude oil is a stable combination of oil and brine. To extract salt water, a significant volume of crude oil is pumped between two highly charged electrodes in Phase I. The water droplets evaporate and are disposed of. When oil has been separated from water, copper oxide is employed to treat it. Reactor I produces copper sulphide precipitate by the reaction of copper oxide and sulfur-containing petroleum. To eliminate this precipitate, phase II requires filtration.

The bulk of the crude oil in the furnace must be heated to 400 degrees Celsius during Phase II. Every component has vanished, with the exception of the asphalt residue cake. The vapor (server II) is passing through the fractionation column. The tower-like fractionation column is tall and cylindrical. Within are multiple horizontal trays made of stainless steel. Each tray has an open chimney that is covered with a loose cap. When it comes to fractionation, the lower tray has high boiling points while the top tray has low boiling values. Fractionation yields uncondensed gas and petroleum products such road tar, diesel oil, heavy oil, naphtha, and kerosene. The renovation process starts immediately in the event that server I or server.

Before departing, the operator must carry out the following tasks: cleaning and checking the tools and etc. When the operator returns to the refining of crude petroleum process, if the quantity of crude oil is less than a batch quantity, he continues to work on other projects until he discovers a sufficient quantity.

The above mentioned process can be described as a $M_X/G(a\ ,b)/1$ queueing system with two stages of heterogeneous service and server breakdown, where arrivals take place according to a Poisson process with arrival rate ƛ. The server is turned on to offer each unit with two phases of heterogeneous service in succession of size min $(\xi\ ,b)$ customers, where $b \geq a$, once it detects at least a customers waiting for service, let's say ξ: Two service phases: the service phase I (FPS) and the service phase II (SPS) (Busy periods). It is expected that the service discipline is FCFS. The assumption is that the service times will follow general laws and a PDF. The service channels will fail for a brief period of time while the server is functioning with the first phase of service, which has a breakdown probability of (π_1) at any moment, and the second phase of service, which has a breakdown probability of (π_2) at any moment. If a server breaks down during any phase of a batch of service, it is instantly sent for repair. The server can begin providing service to the remaining clients after it has been repaired. The server executes closedown work at its closedown time (C) once each SPS is finished. If the queue length is smaller than a the server then departs for a number of random-length vacations. Otherwise continues to serve for the next batch.

2.1. Notations and assumptions

This work employs the following notations.

ƛ is the arrival rate, let μ_1 and μ_1 represent the service rate during peak periods in phases I and II respectively, $X(z_1)$ is the PGF of X, which is the group size random variable, and is the probability that $X = k$. Assume that S1(.), S2(.), R1(.), R2(.), and V(.) represent the CDF of service time in phase I, services time in phase II, renovation time in phase I, renovation time in phase II, and vacation time, respectively, define ${S^0}_1(t_1)$, ${S^0}_2(t_1)$, ${R^0}_1(t_1)$, ${R^0}_2(t_1)$ and $V^0(t_1)$ as the remaining service time in regular period in phase I of a batch time at time 't_1', remaining service time in regular period in phase II of a batch time at time t_1, remaining renovation time in phase I and phase II respectively and denote $\tilde{S}_1(\theta_1)$, $\tilde{S}_2(\theta_1)$, $\tilde{R}_1(\theta_1)$, $\tilde{R}_2(\theta_1)$, and $\tilde{V}(\theta_1)$the LST of S_1, S_2, R_1, R_2 and V respectively.

$N_{q_1}(t_1)$ – size of the queue at time t_1.

$N_{s_1}(t_1)$ – Customers using the service at the moment t_1.

$Y_1(t_1) = 0$ – whenever the server is away.

$Y_1(t_1) = 1$ if the server is performing phase I service while being busy.

$Y_1(t_1) = 2$ if the server is performing phase II service while being busy.

$Y_1(t_1) = 3$ if the server is undergoing the initial step of renovation.

$Y_1(t_1) = 4$ if the server is undergoing the final step of renovation.
$Y_1(t_1) = 5$ if the server is performing a shutdown task.
$Z_1(t_1) = j$ if the server is taking a vacation, it will begin during the idle period.

$P_{i\,j}^{(1)}(x_1, t_1)dt_1 = \Pr\{N_{s_1}(t_1) = i, N_{q_1}(t_1) = j, x_1 \le S_1^0(x_1) \le x_1 + dt_1, Y_1(t_1) = 1\}$,
$(a \le i \le b), (j \ge 0)$,
$P_{i\,j}^{(2)}(x_1, t_1)dt_1 = \Pr\{N_{s_2}(t_1) = i, N_{q_1}(t_1) = j, x_1 \le S_2^0(x_1) \le x_1 + dt_1, Y_1(t_1) = 2\}$,
$(j \ge 0)$,
$Q_{j\,n}(x_1, t_1)dt_1 = \Pr\{N_{q_1}(t_1) = n, x_1 \le V^0(t_1) \le x_1 + dt_1, Y_1(t_1) = 0, Z_1(t_1) = j\}$,
$(n \ge 0), (j \ge 1)$,
$R_n^{(1)}(x_1, t_1)dt_1 = \Pr\{N_{q_1}(t_1) = n, x_1 \le R_1^0(t_1) \le x_1 + dt_1, Y_1(t_1) = 3\}, \quad n \ge 0$,
$R_n^{(2)}(x_1, t_1)dt_1 = \Pr\{N_{q_1}(t_1) = n, x_1 \le R_2^0(t_1) \le x_1 + dt_1, Y_1(t_1) = 4\}, \quad n \ge 0$,
$C_n(x_1, t_1)dt_1 = \Pr\{N_{q_1}(t_1) = n, x_1 \le C^0(t_1) \le x_1 + dt_1, Y_1(t_1) = 5\}, \quad n \ge 0.$

2.2. Steady state equations

Using the supplementary variable technique, the queueing system's equations are obtained as follows:

$$P_{i\,0}^{(1)}(x_1 - \Delta t_1, t_1 + \Delta t_1) = P_{i\,0}^{(1)}(x_1, t_1)(1 - \lambda \Delta t_1) + \sum_{m=a}^{b} P_{mi}^{(1)}(0, t_1)s_1(x_1)\Delta t_1 + R_i^{(1)}(0, t_1)s_1(x_1)\Delta t_1 + \sum_{i=1}^{\infty} Q_{1i}(0)s_1(x_1)\Delta t_1, \quad a \le i \le b, \tag{1}$$

$$P_{ij}^{(1)}(x_1 - \Delta t_1, t_1 + \Delta t_1) = P_{ij}^{(1)}(x_1, t_1)(1 - \lambda \Delta t_1) + \sum_{k=1}^{j} P_{ij-k}^{(1)}(x_1, t_1)\lambda g_k \Delta t_1, \ \ a \le i \le b-1, j \ge 1), \tag{2}$$

$$P_{bj}^{(1)}(x_1 - \Delta t_1, t_1 + \Delta t_1) = P_{bj}^{(1)}(x_1, t_1)(1 - \lambda \Delta t_1) + \sum_{m=a}^{b} P_{m\,b+j}^{(1)}(0, t_1)s_1(x_1)\Delta t_1 + R_{b+j}^{(1)}(0, t_1)s_1(x_1)\Delta t_1 + \sum_{k=1}^{j} P_{bj-k}^{(1)}(x_1, t_1)\lambda g_k \Delta t_1 + \sum_{j=1}^{\infty} Q_{1b+j}(0)s_1(x_1)\Delta t_1, \quad j \ge 1, \tag{3}$$

$$P_{i\,0}^{(2)}(x_1 - \Delta t_1, t_1 + \Delta t_1) = P_{i\,0}^{(2)}(x_1, t_1)(1 - \lambda \Delta t_1) + \sum_{m=a}^{b} P_{mi}^{(2)}(0, t_1)s_2(x_1)\Delta t_1 + P_{i0}^{(1)}(0, t_1)s_2(x_1)\Delta t_1 + R_0^{(2)}(0, t_1)s_1(x_1)\Delta t_1, \quad a \le i \le b, \tag{4}$$

$$P_{ij}^{(2)}(x_1 - \Delta t_1, t_1 + \Delta t_1) = P_{ij}^{(2)}(x_1, t_1)(1 - \lambda \Delta t_1) + \sum_{k=1}^{j} P_{ij-k}^{(2)}(x_1, t_1)\lambda g_k \Delta t_1, \quad a \le i \le b-1, \quad j \ge 1, \tag{5}$$

$$P_{bj}^{(2)}(x_1 - \Delta t_1, t_1 + \Delta t_1) = P_{bj}^{(2)}(x_1, t_1)(1 - \lambda \Delta t_1) + \sum_{m=a}^{b} P_{m\,b+j}^{(2)}(0, t_1)s_2(x_1)\Delta t_1 + R_{b+j}^{(2)}(0, t_1)s_2(x_1)\Delta t_1 + \sum_{k=1}^{j} P_{bj-k}^{(2)}(x_1, t_1)\lambda g_k \Delta t_1, \tag{6}$$
$j \ge 1$,

$$C_n(x_1 - \Delta t_1, t_1 + \Delta t_1) = C_n(x_1, t_1)(1 - \lambda\Delta t_1) + \sum_{m=a}^{b} P_{mn}^{(2)}(0, t_1)c(x_1)\Delta t_1 + \sum_{k=1}^{n} C_{n-k}(x_1, t_1)\lambda g_k \Delta t_1, \quad n \leq a - 1, \tag{7}$$

$$C_n(x_1 - \Delta t_1, t_1 + \Delta t_1) = C_n(x_1, t_1)(1 - \lambda\Delta t_1) + \sum_{k=1}^{n} C_{n-k}(x_1, t_1)\lambda g_k \Delta t_1, \quad n \geq a, \tag{8}$$

$$Q_{10}(x_1 - \Delta t_1, t_1 + \Delta t_1) = Q_{10}(x_1, t_1)(1 - \lambda\Delta t_1) + C_0(0)v(x_1)\Delta t_1, \tag{9}$$

$$Q_{1n}(x_1 - \Delta t_1, t_1 + \Delta t_1) = Q_{1n}(x_1, t_1)(1 - \lambda\Delta t_1) + C_n(0)v(x_1)\Delta t_1 + \sum_{k=1}^{n} Q_{1,n-k}(x_1, t_1)\lambda g_k \Delta t_1, \quad n \geq 1, \tag{10}$$

$$Q_{j0}(x_1 - \Delta t_1, t_1 + \Delta t_1) = Q_{j0}(x_1, t_1)(1 - \lambda\Delta t_1) + Q_{j-1,0}(0, t_1)v(x_1)\Delta t_1, \quad j \geq 2, \tag{11}$$

$$Q_{jn}(x_1 - \Delta t_1, t_1 + \Delta t_1) = Q_{jn}(x_1, t_1)(1 - \lambda\Delta t_1) + \sum_{k=1}^{n} Q_{j,n-k}(x_1, t_1)\lambda g_k \Delta t_1 + Q_{j-1,n}(0, t_1)v(x_1)\Delta t_1, \quad 1 \leq n \leq a - 1, \ j \geq 2, \tag{12}$$

$$Q_{jn}(x_1 - \Delta t_1, t_1 + \Delta t_1) = Q_{jn}(x_1, t_1)(1 - \lambda\Delta t_1) + \sum_{k=1}^{n} Q_{j,n-k}(x_1, t_1)\lambda\Delta t_1, \quad n \geq a, \ j \geq 2, \tag{13}$$

$$R_i^{(1)}(x_1 - \Delta t_1, t_1 + \Delta t_1) = R_i^{(1)}(x_1, t_1)(1 - \lambda\Delta t_1) + \pi_1 r_1(x_1) \int_0^\infty P_{i0}^{(1)}(y)dy\Delta t_1, \quad a \leq i \leq b, \tag{14}$$

$$R_{i+j}^{(1)}(x_1 - \Delta t_1, t_1 + \Delta t_1) = R_{i+j}^{(1)}(x_1, t_1)(1 - \lambda\Delta t_1) + \pi_1 r_1(x_1) \int_0^\infty P_{ij}^{(1)}(y)dy\Delta t_1 + \lambda \sum_{k=1}^{j} R_{i+j-k}^{(1)}(x_1, t_1)g_k \Delta t_1, \quad a \leq i \leq b, \ j \geq 1, \tag{15}$$

$$R_i^{(2)}(x_1 - \Delta t_1, t_1 + \Delta t_1) = R_i^{(2)}(x_1, t_1)(1 - \lambda\Delta t_1) + \pi_1 r_1(x_1) \int_0^\infty P_{i0}^{(2)}(y)dy\Delta t_1, \quad a \leq i \leq b, \tag{16}$$

$$R_{i+j}^{(2)}(x_1 - \Delta t_1, t_1 + \Delta t_1) = R_{i+j}^{(2)}(x_1, t_1)(1 - \lambda\Delta t_1) + \pi_1 r_1(x_1) \int_0^\infty P_{ij}^{(2)}(y)dy\Delta t_1 + \lambda \sum_{k=1}^{j} R_{i+j-k}^{(2)}(x_1, t_1)g_k \Delta t_1, \quad a \leq i \leq b, \ j \geq 1. \tag{17}$$

2.3. Steady state equations

Using the supplementary variable technique, the queueing system's equations are obtained as follows:

$$-P_{i\,0}^{(1)'}(x_1) = -\lambda P_{i0}^{(1)}(x_1) + \sum_{m=a}^{b} P_{mi}^{(1)}(0)s_1(x_1) + R_i^{(1)}(0)s_1(x_1) + \sum_{i=1}^{\infty} Q_{1i}(0)s_1(x_1), \tag{18}$$

$a \leq i \leq b$,

$$-P_{ij}^{(1)'}(x_1) = -\lambda P_{ij}^{(1)}(x_1) + \sum_{k=1}^{j} P_{ij-k}^{(1)}(x_1)\lambda g_k, \quad a \leq i \leq b - 1, \quad j \geq 1, \tag{19}$$

$$-P_{bj}^{(1)'}(x_1) = -\lambda P_{bj}^{(1)}(x_1) + \sum_{m=a}^{b} P_{m\,b+j}^{(1)}(0)s_1(x_1) + \sum_{k=1}^{j} P_{bj-k}^{(1)}(x_1)\lambda g_k + R_{b+j}^{(1)}(0)s_1(x_1) + \sum_{j=1}^{\infty} Q_{1b+j}(0)s_1(x_1), \quad j \geq 1, \tag{20}$$

$$-P_{i0}^{(2)'}(x_1) = -\lambda P_{i0}^{(2)}(x_1) + \sum_{m=a}^{b} P_{mi}^{(2)}(0)s_2(x_1) + P_{i0}^{(1)}(0)s_2(x_1) + R_i^{(2)}(0)s_2(x_1), \tag{21}$$

$a \leq i \leq b$,

$$-P_{ij}^{(2)'}(x_1) = -\lambda P_{ij}^{(2)}(x_1) + \sum_{k=1}^{j} P_{mj-k}^{(2)}(x_1)\lambda g_k, \quad a \leq i \leq b-1, \quad j \geq 1, \tag{22}$$

$$-P_{bj}^{(2)'}(x_1) = -\lambda P_{bj}^{(2)}(x_1) + \sum_{m=a}^{b} P_{mb+j}^{(2)}(0)s_2(x_1) + \sum_{k=1}^{j} P_{bj-k}^{(2)}(x_1)\lambda g_k + R_{b+j}^{(2)}(0)s_2(x_1), \quad j \geq 1, \tag{23}$$

$$-C_n'(x_1) = -\lambda C_n(x_1) + \sum_{m=a}^{b} P_{mn}^{(2)}(0,t)C(x_1) + \sum_{k=1}^{n} C_{n-k}(x_1,t_1)\lambda g_k, \quad n \leq a-1, \tag{24}$$

$$-C_n'(x_1) = -\lambda C_n(x_1) + \sum_{k=1}^{n} C_{n-k}(x_1,t_1)\lambda g_k, \quad n \geq a, \tag{25}$$

$$-Q_{10}'(x_1) = -\lambda Q_{10}(x_1) + C_0(0)v(x_1), \tag{26}$$

$$-Q_{1n}'(x_1) = -\lambda Q_{1n}(x_1) + \sum_{k=1}^{n} Q_{1n-k}(x_1)\lambda g_k + C_n(0)v(x_1), \quad n \geq 1, \tag{27}$$

$$-Q_{j0}'(x_1) = -\lambda Q_{j0}(x_1) + Q_{j-1,0}(0)v(x_1), \quad j \geq 2, \tag{28}$$

$$-Q_{jn}'(x_1) = -\lambda Q_{jn}(x_1) + Q_{j-1,0}(0)v(x_1) + \sum_{k=1}^{n} Q_{jn-k}(x_1)\lambda g_k, \quad n \geq a-1, \tag{29}$$

$$-Q_{jn}'(x_1) = -\lambda Q_{jn}(x_1) + \sum_{k=1}^{n} Q_{jn-k}(x_1)\lambda g_k, \quad j \geq 2, \quad n \geq a, \tag{30}$$

$$-R_i^{(1)'}(x_1) = -\lambda R_i^{(1)}(x_1) + \pi_1 r_1(x_1)\int_0^{\infty} P_{i0}^{(1)}(y)dy, \quad 1 \leq i \leq b, \tag{31}$$

$$-R_{i+j}^{(1)'}(x_1) = -\lambda R_{i+j}^{(1)}(x_1) + \pi_1 r_1(x_1)\int_0^{\infty} P_{ij\,0}^{(1)}(y)dy + \lambda\sum_{k=1}^{j} R_{i+j-k}^{(1)}(x_1)g_k, \quad j \leq 1, \tag{32}$$

$$-R_i^{(2)'}(x_1) = -\lambda R_i^{(2)}(x_1) + \pi_1 r_1(x_1)\int_0^{\infty} P_{i0}^{(2)}(y)dy, \quad 1 \leq i \leq b, \tag{33}$$

$$-R_{i+j}^{(2)'}(x_1) = -\lambda R_{i+j}^{(2)}(x_1) + \pi_1 r_1(x_1)\int_0^{\infty} P_{ij\,0}^{(2)}(y)dy + \lambda\sum_{k=1}^{j} R_{i+j-k}^{(2)}(x_1)g_k, \quad j \geq 1. \tag{34}$$

Taking Laplace-Stieltjes transforms on both sides of the Eqs. (18-34) we get:

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$$\theta_1 \tilde{P}_{i0}^{(1)}(\theta_1) - P_{i0}^{(1)}(0) = \lambda \tilde{P}_{i0}^{(1)}(\theta_1) - \sum_{m=a}^{b} P_{mi}^{(1)}(0)\, \tilde{S}_1(\theta_1) - R_i^{(1)}(0)\, \tilde{S}_1(\theta_1) - \sum_{i=1}^{\infty} Q_{1i}(0)\, \tilde{S}_1(\theta_1), \quad a \le i \le b, \tag{35}$$

$$\theta_1 \tilde{P}_{ij}^{(1)}(\theta_1) - P_{ij}^{(1)}(0) = \lambda \tilde{P}_{ij}^{(1)}(\theta_1) - \sum_{k=1}^{j} \tilde{P}_{ij-k}(\theta_1)\lambda g_k, \ a \le i \le b-1, \quad j \ge 1, \tag{36}$$

$$\theta_1 \tilde{P}_{bj}^{(1)}(\theta_1) - P_{bj}^{(1)}(0) = \lambda \tilde{P}_{bj}^{(1)}(\theta_1) - \sum_{m=a}^{b} P_{mb+j}^{(1)}(0)\, \tilde{S}_1(\theta_1) - R_{b+j}^{(1)}(0)\, \tilde{S}_1(\theta_1) - \sum_{j=1}^{\infty} Q_{1b+j}(0)\, \tilde{S}_1(\theta_1), \quad a \le i \le b, \tag{37}$$

$$\theta_1 \tilde{P}_{i0}^{(2)}(\theta_1) - P_{i0}^{(2)}(0) = \lambda \tilde{P}_{i0}^{(2)}(\theta_1) - \sum_{m=a}^{b} P_{mi}^{(2)}(0)\, \tilde{S}_2(\theta_1) - P_{i0}^{(1)}(0)\, \tilde{S}_2(\theta_1) - R_i^{(2)}(0)\, \tilde{S}_2(\theta_1), \quad a \le i \le b, \tag{38}$$

$$\theta_1 \tilde{P}_{ij}^{(2)}(\theta_1) - P_{ij}^{(2)}(0) = \lambda \tilde{P}_{ij}^{(2)}(\theta_1) - \sum_{k=1}^{j} \tilde{P}_{ij-k}(\theta_1)\lambda g_k, \quad a \le i \le b-1, \quad j \ge 1, \tag{39}$$

$$\theta_1 \tilde{P}_{bj}^{(2)}(\theta_1) - P_{bj}^{(2)}(0) = \lambda \tilde{P}_{bj}^{(2)}(\theta_1) - \sum_{m=a}^{b} P_{mb+j}^{(2)}(0)\, \tilde{S}_2(\theta_1) - \sum_{k=1}^{j} \tilde{P}_{bj-k}(\theta_1)\lambda g_k - R_{b+j}^{(2)}(0)\, \tilde{S}_2(\theta_1), \quad a \le i \le b, \tag{40}$$

$$\theta_1 \tilde{C}_n(\theta_1) - \tilde{C}_n(0) = \lambda \tilde{C}_n(\theta_1) + \sum_{m=a}^{b} P_{mn}^{(2)}(0)\tilde{C}(\theta_1) + \sum_{k=1}^{j} C_{n-k}(x_1)\lambda g_k, \quad n \le a-1, \tag{41}$$

$$\theta_1 \tilde{C}_n(\theta_1) - \tilde{C}_n(0) = \lambda \tilde{C}_n(\theta_1) + \sum_{k=1}^{n} C_{n-k}(x_1)\lambda g_k, \quad n \ge a, \tag{42}$$

$$\theta_1 \tilde{Q}_{10}(\theta_1) - Q_{10}(0) = \lambda \tilde{Q}_{10}(\theta_1) - C_0(0)\tilde{V}(\theta_1), \tag{43}$$

$$\theta_1 \tilde{Q}_{1n}(\theta_1) - Q_{1n}(0) = \lambda \tilde{Q}_{10}(\theta_1) - \lambda \sum_{k=1}^{n} \tilde{Q}_{1n-k}(\theta_1) g_k - C_n(0)\tilde{V}(\theta_1), \quad n \ge 1, \tag{44}$$

$$\theta_1 \tilde{Q}_{j0}(\theta_1) - Q_{j0}(0) = \lambda \tilde{Q}_{j0}(\theta_1) - Q_{j-10}(0)\tilde{V}(\theta_1), \quad j \ge 2, \tag{45}$$

$$\theta_1 \tilde{Q}_{jn}(\theta_1) - Q_{jn}(0) = \lambda \tilde{Q}_{jn}(\theta_1) - Q_{j-1n}(0)\tilde{V}(\theta_1) - \sum_{k=1}^{n} \tilde{Q}_{jn-k}(\theta_1) g_k, \quad n \le a-1, \quad j \ge 2, \tag{46}$$

$$\theta_1 \tilde{Q}_{jn}(\theta_1) - Q_{jn}(0) = \lambda \tilde{Q}_{jn}(\theta_1) - \sum_{k=1}^{n} \tilde{Q}_{jn-k}(\theta_1) g_k, \quad n \ge a, \quad j \ge 2, \tag{47}$$

$$\theta_1 \tilde{R}_i^{(1)}(\theta_1) - R_i^{(1)}(0) = \lambda \tilde{R}_i^{(1)}(\theta_1) + \pi_1 \tilde{R}_1(\theta_1) \int_0^{\infty} P_{i0}^{(1)}(y)dy, \quad 1 \le i \le b, \tag{48}$$

$$\theta_1 \tilde{R}_{i+j}^{(1)}(\theta_1) - R_{i+j}^{(1)}(0) = \lambda \tilde{R}_{i+j}^{(1)}(\theta_1) + \pi_1 \tilde{R}_1(\theta_1) \int_0^{\infty} P_{ij}^{(1)}(y)dy + \lambda \sum_{k=1}^{n} R_{i+j-k}^{(1)}(\theta_1) g_k, \quad j \ge 1, \tag{49}$$

$$\theta_1 \tilde{R}_i^{(2)}(\theta_1) - R_i^{(2)}(0) = \lambda \tilde{R}_i^{(2)}(\theta_1) + \pi_1 \tilde{R}_1(\theta_1) \int_0^{\infty} P_{i0}^{(2)}(y)dy, \quad 1 \le i \le b, \tag{50}$$

$$\theta_1 \tilde{R}^{(2)}_{i+j}(\theta_1) - R^{(2)}_{i+j}(0) = \lambda \tilde{R}^{(2)}_{i+j}(\theta_1) + \pi_1 \tilde{R}_1(\theta_1) \int_0^\infty P^{(2)}_{ij}(y)dy + \lambda \sum_{k=1}^{n} R^{(1)}_{i+j-k}(\theta_1) g_k, \quad j \geq 1. \tag{51}$$

3. Queue size distribution

Let's define the PGF as follows to determine the queue size distribution:

$$\begin{aligned}
&\tilde{P}^{(1)}_i(z_1,\theta_1) = \sum_{j=0}^{\infty} \tilde{P}^{(1)}_{ij}(\theta_1) z_1^n, \qquad P^{(1)}_i(z_1,0) = \sum_{j=0}^{\infty} P^{(1)}_{ij}(0) z_1^n, \quad a \leq i \leq b, \\
&\tilde{P}^{(2)}_i(z_1,\theta_1) = \sum_{j=0}^{\infty} \tilde{P}^{(2)}_{ij}(\theta_1) z_1^n, \qquad P^{(2)}_i(z_1,0) = \sum_{j=0}^{\infty} P^{(2)}_{ij}(0) z_1^n, \quad a \leq i \leq b, \\
&\tilde{Q}_j(z_1,\theta_1) = \sum_{j=1}^{\infty} \tilde{Q}_{1j}(\theta_1) z_1^n, \qquad Q_j(z_1,0) = \sum_{j=1}^{\infty} Q_{1j}(0) z_1^n, \quad j \geq 1, \\
&\tilde{C}(z_1,\theta_1) = \sum_{n=0}^{\infty} \tilde{C}_n(\theta_1) z_1^n, \qquad C(z_1,0) = \sum_{n=0}^{\infty} C_n(0) z_1^n, \\
&\tilde{R}_i(z_1,\theta_1) = \sum_{n=a}^{\infty} \tilde{R}_n(\theta_1) z_1^n, \qquad R_i(z_1,0) = \sum_{n=a}^{\infty} R_n(0) z_1^n.
\end{aligned} \tag{52}$$

By multiplying the Eqs. (35-51) with suitable power of z_1^n and summing over n, ($n = 0$ to ∞) and using Eq. (52):

$$(\theta_1 - \lambda - \lambda X(z_1))\tilde{Q}_1(z_1,\theta_1) = Q_1(z_1,0) - \tilde{V}(\theta_1) C(z_1,0), \tag{53}$$

$$(\theta_1 - \lambda + \lambda X(z_1))\tilde{Q}_j(z_1,\theta_1) = Q_j(z_1,0) - \tilde{V}(\theta_1) \sum_{n=0}^{a-1} Q_{j-1n}(0) z_1^n, \quad j \geq 2, \tag{54}$$

$$(\theta_1 - \lambda + \lambda X(z_1))\tilde{C}(z_1,\theta_1) = C(z_1,0) - \tilde{C}(\theta_1) \sum_{n=0}^{a-1} \sum_{m=a}^{b} P^{(2)}_{mn}(0) z_1^n, \tag{55}$$

$$(\theta_1 - \lambda + \lambda X(z_1))\tilde{P}^{(1)}_i(z_1,\theta_1) = P^{(1)}_i(z_1,0) - \tilde{S}_1(\theta_1) \sum_{m=a}^{b} P^{(1)}_{mi}(0) - \tilde{S}_1(\theta_1) R^{(1)}_i(0) - \tilde{S}_1(\theta_1) \sum_{i=1}^{\infty} Q_{1i}(0), \quad a \leq i \leq b-1 \tag{56}$$

$$\begin{aligned}
(\theta_1 - \lambda + \lambda X(z_1))\tilde{P}^{(1)}_b(z_1,\theta_1) = P^{(1)}_b(z_1,0) \\
- \frac{\tilde{S}_1(\theta_1)}{z_1{}^b} \Bigg\{ \sum_{m=a}^{b} \left(P^{(1)}_m(z_1,0) - \sum_{j=0}^{b-1} P^{(1)}_{mj}(0)\, z_1{}^j \right) + \sum_{i=1}^{\infty} \left(Q_1(z_1,0) - \sum_{j=0}^{b-1} Q_{1j}(0)\, z_1{}^j \right) \\
+ \left(R^{(1)}(z_1,0) - \sum_{n=a}^{b-1} R^{(1)}_n(0)\, z_1{}^n \right) \Bigg\},
\end{aligned} \tag{57}$$

$$(\theta_1 - \lambda + \lambda X(z_1))\tilde{P}^{(2)}_i(z_1,\theta_1) = P^{(2)}_i(z_1,0) - \tilde{S}_2(\theta_1) P^{(1)}_{i0}(0) - \tilde{S}_2(\theta_1) \sum_{m=a}^{b} P^{(2)}_{mi}(0) - \tilde{S}_2(\theta_1) R^{(2)}_i(0), \quad a \leq i \leq b-1, \tag{58}$$

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$$\begin{aligned}(\theta_1 - \lambda + \lambda X(z_1))\tilde{P}_b^{(2)}(z_1, \theta_1) &= P_b^{(2)}(z_1, 0) \\ &- \frac{\tilde{S}_2(\theta_1)}{{z_1}^b}\left\{\sum_{m=a}^{b}\left(P_m^{(2)}(z_1, 0) - \sum_{j=0}^{b-1} P_{mj}^{(2)}(0)\, {z_1}^j\right) + P_{b0}^{(2)}(0)\right. \\ &\left. + \sum_{i=1}^{\infty}\left(R^{(2)}(z_1, 0) - \sum_{n=a}^{b-1} R_n^{(2)}(0)\, {z_1}^n\right)\right\}\end{aligned} \tag{59}$$

$$(\theta_1 - \lambda + \lambda X(z_1))\tilde{R}^{(1)}(z_1, \theta_1) = R^{(1)}(z_1, 0) - \pi_1 \tilde{R}_1(\theta_1) \sum_{i=a}^{b} \tilde{P}_i^{(1)}(z_1, 0), \tag{60}$$

$$(\theta_1 - \lambda + \lambda X(z_1))\tilde{R}^{(2)}(z_1, \theta_1) = R^{(2)}(z_1, 0) - \pi_2 \tilde{R}_2(\theta_1) \sum_{i=a}^{b} \tilde{P}_i^{(2)}(z_1, 0). \tag{61}$$

By substituting $\theta_1 = \lambda - \lambda X(z_1)$ in the Eqs. (53-61), we get:

$$Q_1(z_1, 0) = \tilde{V}(\lambda - \lambda X(z_1)) C(z_1, 0), \tag{62}$$

$$Q_j(z_1, 0) = \tilde{V}(\lambda - \lambda X(z_1)) \sum_{n=0}^{a-1} Q_{j-1n}(0) z_1^n, \quad j \geq 2, \tag{63}$$

$$C(z_1, 0) = \tilde{C}(\lambda - \lambda X(z_1)) \left[\sum_{n=0}^{a-1} \sum_{m=a}^{b} P_{mn}^{(2)}(0) z_1^n\right], \tag{64}$$

$$P_i^{(1)}(z_1, 0) = \tilde{S}_1(\lambda - \lambda X(z_1)) \left[\sum_{m=a}^{b} P_{mi}^{(1)}(0) + R_i^{(1)}(0) + \sum_{i=1}^{\infty} Q_{1i}(0)\right], \quad a \leq i \leq b-1, \tag{65}$$

$$P_b^{(1)}(z_1, 0) = \frac{\tilde{S}_1(\lambda - \lambda X(z_1)) f_1(z_1)}{{z_1}^b - \tilde{S}_1(\lambda - \lambda X(z_1))}, \tag{66}$$

where:

$$f_1(z) = \sum_{m=a}^{b-1} P_m^{(1)}(z, 0) + \sum_{l=1}^{\infty} Q_l(z, 0) + R^{(1)}(z, 0) - \left[\sum_{i=0}^{b-1}\left(p_i^{(1)} + R_i^{(1)}\right) z^i + \sum_{i=0}^{b-1} q_i z^i\right],$$

$$P_i^{(2)}(z_1, 0) = \tilde{S}_2(\lambda - \lambda X(z_1)) \left[\sum_{m=a}^{b} P_{mi}^{(2)}(0) + R_i^{(2)}(0) + P_{i0}^{(2)}(0)\right], \quad a \leq i \leq b-1, \tag{67}$$

$$P_b^{(2)}(z_1, 0) = \frac{\tilde{S}_1(\lambda - \lambda X(z_1)) f_2(z_1)}{{z_1}^b - \tilde{S}_1(\lambda - \lambda X(z_1))}, \tag{68}$$

where:

$$f_2(z) = \sum_{m=a}^{b-1} P_m^{(2)}(z, 0) + R^{(2)}(z, 0) - \left[\sum_{i=0}^{b-1}\left(p_i^{(2)} + R_i^{(2)}\right) z^i\right],$$

$$R^{(1)}(z_1, 0) = \pi_1 \tilde{R}_1(\lambda - \lambda X(z_1)) \sum_{i=a}^{b} \tilde{P}_i^{(1)}(z_1, 0), \tag{69}$$

$$R^{(2)}(z_1, 0) = \pi_1 \tilde{R}_2(\lambda - \lambda X(z_1)) \sum_{i=a}^{b} \tilde{P}_i^{(2)}(z_1, 0), \tag{70}$$

here:

$$p_i^{(1)} = \sum_{m=a}^{b} P_{mi}^{(1)}(0), \quad p_i^{(2)} = \sum_{m=a}^{b} P_{mi}^{(2)}(0), \quad q_i = \sum_{i=1}^{\infty} Q_{1i}(0),$$
$$R_i^{(2)} = R_i^{(2)}(0), \quad R_i^{(2)} = R_i^{(2)}(0).$$

Using the Eqs. (62-70) in (53-61), after simplification we get:

$$\tilde{Q}_1(z_1,\theta_1) = \frac{[\tilde{V}(\lambda - \lambda X(z_1)) - \tilde{V}(\theta_1)]C(z_1,\theta_1)}{(\theta_1 - \lambda + \lambda X(z_1))}, \tag{71}$$

$$\tilde{Q}_j(z_1,\theta_1) = \frac{\left[\tilde{V}\left(\lambda - \lambda X(z_1)\right) - \tilde{V}(\theta)\right]\sum_{n=0}^{a-1} Q_{j-1n}(0) z_1^n}{\left(\theta_1 - \lambda + \lambda X(z_1)\right)}, \quad j \geq 2, \tag{72}$$

$$\tilde{C}(z_1,0) = \frac{\left[\tilde{C}\left(\lambda - \lambda X(z_1)\right) - \tilde{C}(\theta_1)\right]\left[\sum_{n=0}^{a-1}\sum_{m=a}^{b} P_{mn}^{(2)}(0) z_1^n\right]}{\left(\theta_1 - \lambda + \lambda X(z_1)\right)}, \tag{73}$$

$$\tilde{P}_i^{(1)}(z_1,0) = \frac{\left[\tilde{S}_1\left(\lambda - \lambda X(z_1)\right) - \tilde{S}_1(\theta_1)\right]\left[\sum_{m=a}^{b} P_{mi}^{(1)}(0) + R_i^{(1)}(0) + \sum_{i=1}^{\infty} Q_{1i}(0)\right]}{\left(\theta_1 - \lambda + \lambda X(z_1)\right)}, \tag{74}$$

$a \leq i \leq b-1,$

$$\tilde{P}_b^{(1)}(z_1,0) = \frac{\left[\tilde{S}_1(\lambda - \lambda X(z_1)) - \tilde{S}_1(\theta_1)\right] f_1(z_1)}{\left(\theta_1 - \lambda + \lambda X(z_1)\right)\left({z_1}^b - \tilde{S}_1(\lambda - \lambda X(z_1))\right)}, \tag{75}$$

$$\tilde{P}_i^{(2)}(z_1,0) = \frac{\left[\tilde{S}_2\left(\lambda - \lambda X(z_1)\right) - \tilde{S}_2(\theta_1)\right]\left[\sum_{m=a}^{b} P_{mi}^{(2)}(0) + R_i^{(2)}(0) + P_0^{(1)}(0)\right]}{\left(\theta_1 - \lambda + \lambda X(z_1)\right)}, \tag{76}$$

$a \leq i \leq b-1,$

$$\tilde{P}_b^{(2)}(z_1,0) = \frac{\left[\tilde{S}_2(\lambda - \lambda X(z_1)) - \tilde{S}_2(\theta_1)\right] f_2(z_1)}{\left(\theta_1 - \lambda + \lambda X(z_1)\right)\left({z_1}^b - \tilde{S}_2(\lambda - \lambda X(z_1))\right)}, \tag{77}$$

$$\tilde{R}_1(z_1,\theta_1) = \frac{[\tilde{R}_1(\lambda - \lambda X(z_1)) - \tilde{R}_1(\theta_1)]\pi_1 \sum_{i=a}^{b} \tilde{P}_i^{(1)}(z_1,0)}{\left(\theta_1 - \lambda + \lambda X(z_1)\right)}, \tag{78}$$

$$\tilde{R}_2(z_1,\theta_1) = \frac{[\tilde{R}_2(\lambda - \lambda X(z_1)) - \tilde{R}_2(\theta_1)]\pi_2 \sum_{i=a}^{b} \tilde{P}_i^{(2)}(z_1,0)}{\left(\theta_1 - \lambda + \lambda X(z_1)\right)}. \tag{79}$$

3.1. PGF of the queue size at different epochs

3.1.1. Close down completion epoch

Using the Eqs. (24) to (25) and substituting $\theta = 0$ and after some algebra, we get:

$$C(z_1) = \frac{\left(\tilde{C}\left(\lambda - \lambda X(z_1)\right) - 1\right)\sum_{i=0}^{a-1} P_i^{(2)} {z_1}^i}{\left(-\lambda + \lambda X(z_1)\right)}. \tag{80}$$

3.1.2. Vacation completion epoch

Using the Eqs. (26) to (30) and substituting $\theta = 0$ and after some computation, we get:

$$V(z_1) = \frac{\left(\tilde{V}\left(\lambda - \lambda X(z_1)\right) - 1\right)\left[\tilde{C}\left(\lambda - \lambda X(z_1)\right)\sum_{i=0}^{a-1} P_i^{(1)} z^i + \sum_{i=0}^{a-1} q_i {z_1}^i\right]}{\left(-\lambda + \lambda X(z_1)\right)}. \tag{81}$$

3.1.3. Service in phase I completion epoch

$$P_1(z_1) = \frac{\left\{\begin{array}{c}\left(\tilde{S}_1(\lambda-\lambda X(z_1))-1\right)\sum_{i=a}^{b-1}\left(P_i^{(1)}(z_1{}^b-z_1{}^i)+R_i^{(1)}(z_1{}^b-z_1{}^i)+q_i(z_1{}^b-z_1{}^i)\right)\\ +\left(\tilde{S}_1(\lambda-\lambda X(z_1))-1\right)\tilde{V}(\lambda-\lambda X(z_1))\tilde{C}\left((\lambda-\lambda X(z_1))-1\right)\sum_{i=0}^{a-1}P_i^{(1)}z_1{}^i\\ +\left(\tilde{S}_1(\lambda-\lambda X(z_1))-1\right)\left(\tilde{V}(\lambda-\lambda X(z_1))-1\right)\sum_{i=0}^{a-1}q_iz_1{}^i\end{array}\right\}}{\left\{\begin{array}{c}\left(\lambda-\lambda X(z_1)\right)\left(1+\pi_1\tilde{R}_1(\lambda-\lambda X(z_1))-\pi_1\tilde{S}_1(\lambda-\lambda X(z_1))\tilde{R}_1(\lambda-\lambda X(z_1))\right)\\ \tilde{R}_1(\lambda-\lambda X(z_1))\left(z_1{}^b-\tilde{S}_1(\lambda-\lambda X(z_1))\right)\end{array}\right\}}. \quad (82)$$

3.1.4. Service in phase II completion epoch

$$P_2(z_1) = \frac{\left(\tilde{S}_2(\lambda-\lambda X(z_1))-1\right)\sum_{i=a}^{b-1}\left(P_i^{(2)}(z_1{}^b-z_1{}^i)+R_i^{(2)}(z_1{}^b-z_1{}^i)+P_0^{(1)}(z_1{}^b-z_1{}^i)\right)}{\left\{\begin{array}{c}\left(\lambda-\lambda X(z_1)\right)\left(1+\pi_2\tilde{R}_2(\lambda-\lambda X(z_1))-\pi_2\tilde{S}_2(\lambda-\lambda X(z_1))\tilde{R}_2(\lambda-\lambda X(z_1))\right)\\ \left(z_1{}^b-\tilde{S}_2(\lambda-\lambda X(z_1))\right)\end{array}\right\}}. \quad (83)$$

3.1.5. Renovation in phase I completion epoch

$$R_1(z_1) = \frac{\left\{\begin{array}{c}\pi_1\left(\tilde{R}_1(\lambda-\lambda X(z_1))-1\right)\left(\tilde{S}_1(\lambda-\lambda X(z_1))-1\right)\\ \cdot\sum_{i=a}^{b-1}\left(P_i^{(1)}(z_1{}^b-z_1{}^i)+R_i^{(1)}(z_1{}^b-z_1{}^i)+q_i(z_1{}^b-z_1{}^i)\right)\\ +\pi_1\left(\tilde{R}_1(\lambda-\lambda X(z_1))-1\right)\left(\tilde{S}_1(\lambda-\lambda X(z_1))-1\right)\\ \cdot\tilde{V}(\lambda-\lambda X(z_1))\tilde{C}\left((\lambda-\lambda X(z_1))-1\right)\sum_{i=0}^{a-1}P_i^{(1)}z_1{}^i\\ +\pi_1\left(\tilde{R}_1(\lambda-\lambda X(z_1))-1\right)\left(\tilde{S}_1(\lambda-\lambda X(z_1))-1\right)\left(\tilde{V}(\lambda-\lambda X(z_1))-1\right)\sum_{i=0}^{a-1}q_iz_1{}^i\end{array}\right\}}{\left\{\begin{array}{c}\left(\lambda-\lambda X(z_1)\right)\left(1+\pi_1\tilde{R}_1(\lambda-\lambda X(z_1))-\pi_1\tilde{S}_1(\lambda-\lambda X(z_1))\tilde{R}_1(\lambda-\lambda X(z_1))\right)\\ \tilde{R}_1(\lambda-\lambda X(z_1))\left(z_1{}^b-\tilde{S}_1(\lambda-\lambda X(z_1))\right)\end{array}\right\}}. \quad (84)$$

3.1.6. Renovation in phase II completion epoch

$$R_2(z_1) = \frac{\left\{\begin{array}{c}\pi_2\left(\tilde{R}_2(\lambda-\lambda X(z_1))-1\right)\left(\tilde{S}_2(\lambda-\lambda X(z_1))-1\right)\\ \sum_{i=a}^{b-1}\left(P_i^{(2)}(z_1{}^b-z_1{}^i)+R_i^{(2)}(z_1{}^b-z_1{}^i)+P_0^{(1)}(z_1{}^b-z_1{}^i)\right)\end{array}\right\}}{\left\{\begin{array}{c}\left(\lambda-\lambda X(z_1)\right)\left(1+\pi_1\tilde{R}_1(\lambda-\lambda X(z_1))-\pi_1\tilde{S}_1(\lambda-\lambda X(z_1))\tilde{R}_1(\lambda-\lambda X(z_1))\right)\\ \tilde{R}_1(\lambda-\lambda X(z_1))\left(z_1{}^b-\tilde{S}_1(\lambda-\lambda X(z_1))\right)\end{array}\right\}}. \quad (85)$$

3.2. PGF of queue size at an arbitrary time epoch

Obtaining the PGF of the queue size at any given time epoch is as follows:

$$P(z_1) = \tilde{C}(z_1,0) + \sum_{m=a}^{b-1} \tilde{P}_m^{(1)}(z_1,0) + \tilde{P}_b^{(1)}(z_1,0) + \sum_{m=a}^{b-1} \tilde{P}_m^{(2)}(z_1,0) + \tilde{P}_b^{(2)}(z_1,0) + \tilde{R}_1(z_1,0) + \tilde{R}_2(z_1,0). \tag{86}$$

Substitute Eqs. (58-66) in Eq. (73) with $\theta = 0$ we get:

$$P(z_1) = \frac{\left\{\begin{array}{c} \left(\tilde{S}_1(\lambda - \lambda X(z_1)) - 1\right)\left[\pi_1\left(\tilde{R}_1(\lambda - \lambda X(z_1)) - 1\right) + 1\right] D_2 \sum_{i=a}^{b-1} C_i^{(1)}({z_1}^b - {z_1}^i) \\ +\left(\tilde{S}_2(\lambda - \lambda X(z_1)) - 1\right)\left[\pi_2\left(\tilde{R}_2(\lambda - \lambda X(z_1)) - 1\right) + 1\right] D_1 \sum_{i=a}^{b-1} C_i^{(2)}({z_1}^b - {z_1}^i) \\ +\left(\tilde{S}_2(\lambda - \lambda X(z_1)) - 1\right)\left[\pi_2\left(\tilde{R}_2(\lambda - \lambda X(z_1)) - 1\right) + 1\right] D_1 \sum_{i=a}^{b-1} P_0^{(1)}({z_1}^b - {z_1}^i) \\ +\left(\tilde{S}_2(\lambda - \lambda X(z_1)) - 1\right)\left[\pi_2\left(\tilde{R}_2(\lambda - \lambda X(z_1)) - 1\right) + 1\right]\tilde{V}(\lambda - \lambda X(z_1)) \\ \tilde{C}\left(\left(\lambda - \lambda X(z_1)\right) - 1\right) D_1 D_2 \sum_{i=0}^{a-1} P_i^{(1)} {z_1}^i + \left(\tilde{S}_1(\lambda - \lambda X(z_1)) - 1\right) \\ \left[\pi_1\left(\tilde{R}_1(\lambda - \lambda X(z_1)) - 1\right) + 1\right]\tilde{V}(\lambda - \lambda X(z_1) - 1) D_2 + \tilde{V}(\lambda - \lambda X(z_1) - 1) D_1 D_2 \\ \sum_{i=0}^{a-1} q_i {z_1}^i + \tilde{C}\left((\lambda - \lambda X(z_1)) - 1\right) D_1 D_2 \sum_{i=0}^{a-1} P_i^{(2)} {z_1}^i \end{array}\right\}}{(-\lambda + \lambda X(z_1)) D_1 D_2}, \tag{87}$$

where:

$$D_1 = \left(1 + \pi_1 \tilde{R}_1(\lambda - \lambda X(z_1)) - \pi_1 \tilde{S}_1(\lambda - \lambda X(z_1)) \tilde{R}_1(\lambda - \lambda X(z_1))\right)\left({z_1}^b - \tilde{S}_1(\lambda - \lambda X(z_1))\right),$$
$$D_2 = \left(1 + \pi_2 \tilde{R}_2(\lambda - \lambda X(z_1)) - \pi_2 \tilde{S}_2(\lambda - \lambda X(z_1)) \tilde{R}_2(\lambda - \lambda X(z_1))\right)\left({z_1}^b - \tilde{S}_2(\lambda - \lambda X(z_1))\right).$$

3.3. Steady state condition

$P(1) = 1$ must be satisfied by the probability generating function. Applying L'Hospital principles and equating the expression to 1 will satisfy this requirement. Consecutively:

$$\begin{aligned} &E(S_1)\left(b - \lambda E(X)E(S_2)\right)\sum_{i=a}^{b-1} C_i^{(1)}({z_1}^b - {z_1}^i) \\ &\quad + E(S_2)\left(b - \lambda E(X)E(S_1)\right)\sum_{i=a}^{b-1} C_i^{(1)}({z_1}^b - {z_1}^i) + E(S_2)\big(b \\ &\quad - \lambda E(X)E(S_1)\big)\sum_{i=a}^{b-1} P_0({z_1}^b - {z_1}^i) + E(S_1)\left(b - \lambda E(X)E(S_2)\right) \\ &\quad + \left(b - \lambda E(X)E(S_1)\right)\left(b - \lambda E(X)E(S_2)\right)[\lambda E(X)E(V) + \lambda E(X)E(C)]\sum_{i=0}^{a-1} P_i^{(1)} + \big(b \\ &\quad - \lambda E(X)E(S_1)\big) + \left(b - \lambda E(X)E(S_2)\right)\ \lambda E(X)E(C)\sum_{i=0}^{a-1} P_i^{(2)} \\ &+E(S_1)\left(b - \lambda E(X)E(S_2)\right) + \left(b - \lambda E(X)E(S_1)\right)\left(b - \lambda E(X)E(S_2)\right)\lambda E(X)E(V)\sum_{i=0}^{a-1} q_i \\ &\quad = \left(b - \lambda E(X)E(S_1)\right)\left(b - \lambda E(X)E(S_2)\right). \end{aligned} \tag{88}$$

As a result $P(1) = 1$ satisfied if:

$$\left(z_1{}^b - \tilde{S}_1\big(\lambda - \lambda X(z_1)\big)\right)\left(z_1{}^b - \tilde{S}_2\big(\lambda - \lambda X(z_1)\big)\right) > 0, \quad \rho = \lambda^2 \frac{\big(E(X)\big)^2 E(S_1)E(S_2)}{b}.$$

Theorem 1.

$$q_n = \sum_{i=0}^{n} K_i \, P_{n-1}^{(2)}, \qquad n = 0, 1, 2, 3, \ldots.. a-1,$$

where $K_n = \frac{h_n + \sum_{i=0}^{n} a_i K_{n-i}}{1-\alpha_0}$, $n = 0, 1, 2, 3, \ldots.. a-1$ with $K_0 = \frac{\alpha_0 \beta_0}{1-\alpha_0}$, $h_n = \sum_{i=0}^{n} \alpha_0 \, \beta_{n-i}$.

The possibilities that a customer will arrive during a holiday or a shut-down period are $\alpha_i' s$ and $\beta_i' s$ respectively.

Proof.

Using the Eqs. (44)-(46), $\sum_{j=1}^{\infty} Q_j(z_1, 0)$ simplifies to:

$$\begin{aligned}\sum_{n=0}^{\infty} q_n z_1{}^n &= \tilde{V}\big(\lambda - \lambda X(z_1)\big)\left[\tilde{C}\big(\lambda - \lambda X(z_1)\big)\sum_{n=0}^{a-1} P_n^{(2)} z_1{}^n \sum_{n=0}^{a-1} q_n z_1{}^n\right] \\ &= \left(\sum_{n=0}^{\infty} \alpha_n z_1{}^n\right)\left[\sum_{j=0}^{\infty} \beta_j z_1{}^j \sum_{n=0}^{a-1} P_n^{(2)} z_1{}^n \right. \\ &\left. + \sum_{n=0}^{a-1} q_n z_1{}^n\right] \sum_{j=0}^{n}\left[\left(\sum_{i=0}^{n-j} \alpha_i \beta_{n-i} p_j{}^{(2)} + \sum_{i=0}^{n} \alpha_{n-i} q_i\right)\right] z_1{}^n.\end{aligned} \tag{89}$$

Equating the coefficients of $z_1{}^n$ on both sides of the above equation for $n = 0$, 1, 2, 3, …, $a-1$ we have:

$$q_n = \sum_{j=0}^{n}\left(\sum_{i=0}^{n-j} \alpha_i \, \beta_{n-i-j}\right) P_j^{(2)} + \sum_{i=0}^{n-j} \alpha_{n-i} \, q_i,$$
$$q_n = \sum_{j=0}^{n}\left(\sum_{i=0}^{n-j} \alpha_i \, \beta_{n-i-j}\right) P_j^{(2)} + \sum_{i=0}^{n} \alpha_{n-i} \, q_i.$$

On solving for q_n , we get:

$$q_n = \frac{\sum_{j=0}^{n}\left(\sum_{i=0}^{n-j} \alpha_i \, \beta_{n-i-j}\right) P_j^{(2)} + \sum_{i=0}^{n-1} \alpha_{n-i} \, q_i}{(1-\alpha_0)}.$$

Coefficient of $P_n^{(2)}$ in q_n is $\frac{\alpha_0 \beta_0}{1-\alpha_0} = K_0$.

Coefficient of $P_{n-1}^{(2)}$ in q_{n-1} is $(h_1 + \alpha_1)$.

Coefficient of $\frac{P_{n-1}^{(2)}}{1-\alpha_0}$ in q_{n-1} is $\frac{h_1 + \alpha_1 K_0}{1-\alpha_0} = K_1$.

3.4. Expected length of busy period

Let B be the random variable for the busy period. Then estimated duration of the busy season is:

$$E(B) = \frac{\left(E(T_1) + E(T_2)\right)}{\sum_{i=0}^{a-1} d_i} + \pi_1 E(R_1), \quad E(T_2) = E(S_2) + \pi_2 E(R_2), \text{ where } E(T_1) = E(S_1).$$

3.5. Expected length of the idle period

According to Arumuganathan and Jeyakumar's method [1], it is obtained.

Theorem 2.

Let I be the random variable. Then the duration of the waiting period as anticipated is given by, $\mathrm{E}(I) = \mathrm{E}(I_1) + \mathrm{E}(C)$:

$$E(I_1) = \frac{E(V)}{1 - \sum_{n=0}^{a-1} Q_{1n}(0)} = \frac{E(V)}{1 - \sum_{n=0}^{a-1} \sum_{i=0}^{n} \left\{\sum_{j=0}^{n-1} \alpha_j \beta_{n-i-j}\right\} P_i^{(1)}},$$

where I_1 is the random variable denoting the "Idle period due to multiple vacation process", $E(C)$ is the expected closedown time.

By theorem the expected idle period $E(I)$ is obtained as $E(I) = E(I_1) + E(C)$.

3.6. Expected queue length at an arbitrary time epoch

$$E(Q) = \frac{\left\{\begin{gathered} ((b - s_{21})^2 H_{11} + (b - s_{21})^2 (b - s_{11})^2 V_{11}) \sum_{i=0}^{a-1} P_i^{(1)} \\ +((b - s_{21})^2 H_{12} - (b - s_{21})^2 (b - s_{11})^2 V_{12}) \sum_{n=0}^{a-1} i P_i^{(1)} \\ +(b - s_{21})^2 (b - s_{11})^2 L_1 \sum_{i=0}^{a-1} P_i^{(2)} + (b - s_{21})^2 (b - s_{11})^2 L_2 \sum_{n=0}^{a-1} i P_i^{(2)} \\ +((b - s_{21})^2 H_{13} + (b - s_{21})^2 (b - s_{11})^2 V_{13}) \sum_{i=0}^{a-1} q_i \\ +((b - s_{21})^2 H_{14} - (b - s_{21})^2 (b - s_{11})^2 V_{14}) \sum_{n=0}^{a-1} i q_i \\ +(b - s_{21})^2 H_{15} \sum_{i=a}^{b-1} (b - i) C_i^{(1)} + (b - s_{21})^2 H_{16} \sum_{i=a}^{b-1} (b(b-1) - i(i-1)) C_i^{(1)} \\ +(b - s_{21})^2 H_{25} \sum_{i=a}^{b-1} (b - i)\left(C_i^{(2)} + P_0\right) \\ +(b - s_{11})^2 H_{26} \sum_{i=a}^{b-1} (b(b-1) - i(i-1))\left(C_i^{(2)} + P_0\right) \end{gathered}\right\}}{2(\lambda X_1)^2 (b - s_{11})^2 (b - s_{21})^2}. \tag{90}$$

$H_{11} = 2\pi_1 T_{11} S_{11} R_{11} (V_1 + C_1) + T_{11} S_{11} (V_2 + C_2 + 2V_1 C_1) - 2T_{11} S_{12} (V_1 + C_1)$,
$H_{12} = T_{11} S_{11} (V_1 + C_1), \quad H_{13} = 2\pi_1 T_{11} S_{11} R_{11} V_1 + T_{11} S_{11} V_2 - 2T_{11} S_{12} V_1$,
$H_{14} = T_{11} S_{11} V_1, \quad H_{15} = 2\pi_1 T_{11} S_{11} R_{11} - 2T_{11} S_{11}$,
$H_{16} = T_{11} S_{11}, \ T_{11} = (\lambda X_1)(b - s_{11})$,
$H_{25} = 2\pi_1 T_{21} S_{21} R_{21} - 2T_{21} S_{22}, \quad H_{26} = T_{21} S_{21}, \quad T_{21} = (\lambda X_1)(b - s_{21})$,
$V_{11} = \left(V_1(\lambda X_2) - 2V_2(\lambda X_1)\right), \quad V_{12} = 2V_1(\lambda X_1), \quad V_{13} = V_1(\lambda X_2), \quad V_{14} = 2V_1(\lambda X_1)$,
$L_1 = \left((\lambda X_1) C_2 - (\lambda X_2) C_1\right), \quad L_1 = 2(\lambda X_1) C_1$.

3.7. Expected waiting time

The Little's formula is used to calculate the expected waiting time as follows:

$$E(W) = \frac{E(Q)}{\lambda E(X)}, \tag{91}$$

where $E(Q)$ is given in Eq. (89).

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4. Conclusions

In this study, examine the behaviour of the server failure without interruption in a queueing system with $M_X/G(a\,,b)/1$ and two phases of heterogeneous service. We obtain the probability generating function of the queue size at any time epoch, Close down completion epoch, Vacation completion epoch, Service in phase I completion epoch, Service in phase II completion epoch, Renovation in phase I completion epoch and Renovation in phase II completion epoch. In future a cost model is discuss with numerical examples.

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Data availability

The datasets generated during and/or analyzed during the current study are available from the corresponding author on reasonable request.

Author contributions

Palaniammal S and Kumar K established the PGF of queue size and various time epoch of the model. Palaniammal S: project administration, supervision, validation. Kumar K: conceptualization, formal analysis, investigation, methodology, visualization, writing – review and editing.

Conflict of interest

The authors declare that they have no conflict of interest.

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