A third-order shear deformation plate bending formulation for thick plates: first principles derivation and applications

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Abstract. A third-order shear deformation plate bending formulation is presented in this study from the first principles. The derivation assumed a displacement field constructed using third-order polynomial function of the transverse (z) coordinate; and made to apriori satisfy the linear three-dimensional (3D) kinematics relations as well as the transverse shear stress free boundary conditions at the top and bottom plate surfaces. The formulation thus has no need for shear stress correction factors of the first-order shear deformation plate theories. The domain equations of equilibrium are obtained as a set of three coupled differential equations in terms of three unknown displacements. The system of coupled equations is solved for simply supported rectangular and square plates subjected to four cases of loading distributions: sinusoidal loading, uniformly distributed loading, linearly distributed loading and point load at the plate center. Navier's double trigonometric series method is used to construct trial solutions for the three displacement functions such that the boundary conditions are satisfied identically. The integration problem is thus reduced to an algebraic problem and is solved for each considered loading. It is found that the present formulation gives exact results for the normal stresses σ_{xx} for sinusoidal and uniformly distributed loads. The study further showed that the results for deflection and stresses agreed with Krishna Murty's higher order shear deformation plate theory results. The present formulation gave accurate results because of the inclusion of transverse normal strain effects in the formulation. The formulation gives a quadratic variation of the transverse shear stresses across the thickness in consonance with the theory of elasticity method.

Keywords: third-order shear deformation plate bending formulation, thick plate, Navier double trigonometric series method, transverse shear stress, transverse normal stress.

Nomenclature

x, y, z	Three-dimensional Cartesian coordinates
x, y	In-plane Cartesian coordinate
z	Transverse coordinate
a	Length of plate (in-plane dimension of plate)
b	Width (in-plane dimension) of plate
h	Thickness of plate
$u(x, y, z)$	Displacement field component in the x direction
$v(x, y, z)$	Displacement field component in the y direction
$w(x, y, z)$	Transverse (z) component of the displacement
$w_0(x, y) = w(x, y, z = 0)$	Transverse (z) component of the displacement of the middle surface ($z = 0$)
$\alpha_x(x, y), \alpha_y(x, y)$	Rotations of the normals to the middle surfaces of the plate about the y and x axes respectively
$\beta_x(x, y), \beta_y(x, y)$	Displacement warping functions
$\varepsilon_{xx}, \varepsilon_{yy}, \varepsilon_{zz}$	Normal strains in the x, y and z directions respectively
$\gamma_{xy}, \gamma_{yz}, \gamma_{xz}$	Shear strains about the xy, yz and xz planes
$\sigma_{xx}, \sigma_{yy}, \sigma_{zz}$	Normal stresses in the x, y and z directions respectively

τ_{xy}, τ_{yz}, τ_{xz}	Shear stresses in the xy, yz and xz planes
E	Young's modulus of elasticity
G	Shear modulus
D	Modulus of flexural rigidity of plate
μ	Poisson's ratio
M_{xx}, M_{yy}	Bending moments in the x and y directions respectively
M_{xy}	Twisting moment
Q_x, Q_y	Shear force
P_x	Higher order internal stress resultant defined using σ_{xx}
P_y	Higher order internal stress resultant defined using σ_{yy}
P_{xy}	Higher order internal stress resultant defined using τ_{xy}
R_x	Higher order internal stress resultant defined using τ_{xz}
R_y	Higher order internal stress resultant defined using τ_{yz}
Π	Total potential energy functional
2D	Two-dimensional
3D	Three-dimensional
R^2	Two-dimensional domain of plate
$q(x,y)$	Distribution of transverse load over the plate domain
δ	Change in (variation in) or, first variation
N_{xx}	Resultant normal in-plane force in x direction
N_{yy}	Resultant normal in-plane force in y direction
N_{xy}	Resultant in-plane shear force
∇^2	Laplacian; Laplacian operator
∇^4	Biharmonic operator
m, n	Integers
A_{mn}	Unknown parameters used to define double trigonometric series for $w(x,y)$
B_{mn}	Unknown parameters used to define double trigonometric series for $\alpha_x(x,y)$
C_{mn}	Unknown parameters used to define double trigonometric series for $\alpha_y(x,y)$
λ_m	Parameter defined in terms of m, π and a
γ_n	Parameter defined in terms of n, π and b
q_{mn}	Double Fourier series coefficients of the load
Q_{mn}	Parameter defined in terms of q_{mn} and d
q_0	Intensity of uniformly distributed load
$\delta(x=\bar{x}, y=\bar{y})$	Dirac delta function
P_0	Point load
S	Parameter defined in terms of a and h
$\bar{w}$	Dimensionless deflection
$\bar{\sigma}_{xx}$, $\bar{\sigma}_{yy}$	Dimensionless normal stresses
$\bar{\tau}_{xy}$	Dimensionless shear stress
$\bar{\tau}_{zx}$	Dimensionless transverse shear stress
CR	Constitutive relations
EE	Equilibrium equations
R_dPT	Reddy plate theory
M_dPT	Mindlin plate theory
FSDPT	First order shear deformation plate theory
HSDPT	Higher order shear deformation plate theory
CPT	Classical plate theory
KPT	Kirchhoff plate theory

1. Introduction

Plates are three-dimensional (3D) structural members characterized by in-plane dimensions and a transverse dimension that is usually smaller than the in-plane dimensions. They are widely used in civil, structural, mechanical, naval, marine engineering to carry static, dynamic or compression loadings. They may be made of homogeneous or non-homogeneous materials, isotropic, orthotropic, laminated or composite materials, and hence can be classified accordingly based on their material makeup. They are also classified based on their thickness relative to the in-plane dimensions as thin, moderately thick or thick plates.

Depending on the nature of applied loadings, plates may be subject to static flexure, dynamic flexure or buckling.

Generally, plate problems are three-dimensional (3D) problems of elasticity theory. However, a rigorous 3D elasticity formulations of plate behaviour entails complicated mathematical problems even for simple considerations of isotropic, homogeneous materials.

Problems of plates have therefore been expressed by simplifying the 3D elasticity formulations to one-dimensional (1D) and two-dimensional (2D) approximations; and this has been the main objective of plate research.

The classical thin plate theory (CPT) commonly called the Kirchhoff plate theory (KPT) assumed the Navier-Kirchhoff hypothesis [1-10].

– orthogonality of the cross-sectional planes to the middle surface prior to flexural deformation and after bending deformation.

– the invariance of thickness during bending deformation.

– the following are the notable advantages of the KPT [1-16]:

– the equation of equilibrium is linear and contains only one unknown function – deflection of the middle surface.

– the internal force resultants are expressible in terms of the transverse deflection function.

– it gives parabolic variation of shear stresses over the thickness in consonance with mechanics of structures.

The main disadvantage of the KPT is the inability to consider transverse shear deformations, thus limiting the scope of applicability to thin plates where transverse shear deformations are ignorable without significant errors [17-22]. Despite the disadvantages, KPT has been found to be satisfactory for thin plates and various methods for solving KPT are found in references [1-22] and [23-24].

Research efforts to develop improved formulations and postulations to consider shear deformation effects and thus extend the scope of plate theory to moderately thick and thick plates led to the derivations by Reissner [25], [26]; Mindlin [27]; Krishna Murty [28], Srinivas and Rao [29], Shimpi and Patel [30]; Sayyad and Shinde [31]; Ghugal and Gajbiye [32], Sayyad and Shinde [33]; Bathini and Reddy [34], [35]; Bathini et al. [36]; Eipakchi and Moshir [37]; Zagaripoor et al. [38], Raissi et al. [39] and Rodrigues et al. [40].

2. Review of previous works

Ike [20], [22] studied Mindlin's first order shear deformable plates. Nwoji et al [21] obtained satisfactory solutions for the flexural analysis of simply supported rectangular Mindlin plates subjected to sinusoidal transverse load distribution using the Navier's double trigonometric series method.

Ike [41] used Fourier series method to find the stresses and deflections in thick beams. Ike et al [42] used least squares method solve the stresses in rectangular plates under parabolic edge loads. Onah et al [43] derived stress function for solving elastostatic problems of thick circular plates.

Onyeka et al [44-46] and Onyeka and Okeke [47] used polynomial displacement function in an energy formulation to solve the flexural problem of thick plate with simply supported, free and clamped boundaries. Their work considered shear deformation and did not need shear correction factors. They used 3D kinematics and constitutive relations in formulating the energy functional and minimization procedure for the equilibrium equations. This solution for center deflections were in error by 2.9 %-3.7 % compared with the exact solution.

Onyeka and Mama [48] and Onyeka et al [49] have used a trigonometric displacement function in an energy functional minimization method to obtain satisfactory solutions respectively for the bending and stability of thick plates.

Rouzegar and Abdoli Sharifpoor [50] developed a finite element formulation based on two-variable refined plate theory (RPT) for the flexural analysis of isotropic and orthotropic plates. The RPT used is applicable to thin and thick plates and gives parabolic transverse shear stress variation across the plate thickness, thus obviating the need for shear correction as boundary conditions are satisfied. They used variational principles to obtain the equilibrium and weak form equations; and considered a 4-node rectangular plate element with six nodal degrees of freedom. They found satisfactory solutions using MATLAB software on the resulting algebraic problem.

Gajbhiye et al [51] used a 5th order shear and normal deformation theory to satisfactorily solve the eigenvalue problem of free vibration of simply supported thick isotropic square plates.

Gajbhiye et al [52] used a quasi-three-dimensional theory that considered shear and normal deformation to solve the bending problem of simply supported sandwich plates. They considered shear stress free boundary conditions, thus obviating the need for stress correction. They used virtual work principle for the domain equations, and Navier's method for satisfactory solutions for sinusoidal and uniform transverse loadings.

Ghugal and Sayyad [53], [54] used a trigonometric shear deformation theory (TSDT) to obtain satisfactory solutions to the elasticity problem of thick laminated plates under transverse loading.

Ghugal and Gajbhiye [32] developed a 5th order shear deformation theory that considered transverse shear deformation and transverse normal strain effects, and applied it to the bending analysis of thick plates. They used virtual work principles to obtain field equations and boundary conditions and Navier's series method to solve the resulting boundary value problem (BVP).

Zargaripoor et al [55] used exact wave propagation approach for the first time to obtain free vibration and buckling solutions for thick rectangular plates modelled using third-order shear deformation plate theory. They considered plates with opposite simply supported edges while the other edges may be clamped or simply supported. They derived the matrices of wave propagation and reflection for the plate problem and by superposition, obtained the characteristic equation, which was solved for the dimensionless frequencies and buckling loads for the different boundary conditions studied.

Kumar et al [56] have used radial basis function based meshfree methods for the analysis of thick plates using higher order shear deformation theory. Makvandi et al [57] studied the behaviour of moderately thick plates under compressive load. Civalek and Ulker [58] used the harmonic differential quadratic method for the flexural analysis of thin isotropic circular plates. Civalek [59] used the discrete singular convolution (DSC) method to solve bending problems of thick rectangular plates. Other seminal works with significant insight to the thick plate problem are found in references [60-83].

In this study a third-order shear deformation plate bending formulation is presented from first principles for the modelling and solution of thick plate bending problems under transverse loadings.

2.1. Novelty of the study

The novelty of the study is the first principle approach adopted in the paper for the formulation of the thick plate bending problem using a third order shear deformation plate theory that satisfies the transverse shear stress boundary conditions. The study presents a systematic study that uses equilibrium method to derive the governing differential equations of equilibrium. The Navier's double trigonometric series method is used in a first principles, systematic way to derive solutions for:

(i) sinusoidal distribution of transverse load,
(ii) uniformly distributed transverse load,
(iii) linearly varying transverse load,
(iv) transverse point load acting at any point on the plate domain.

2.2. Theoretical framework

The thick plate considered which is shown in Fig. 1 has length a, width b, and thickness h, and is defined using the Cartesian coordinates by: $0 \leq x \leq a$, $0 \leq y \leq b$, $-h/2 \leq z \leq h/2$.

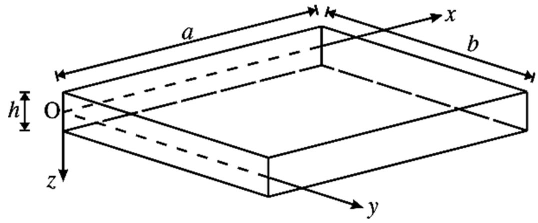

Fig. 1. Thick plate

Basic assumptions of the formulation

(i) The plate material is homogeneous, isotropic and linearly elastic.

(ii The plate is only subjected to transverse load.

(iii) The body forces are ignored, but can be incorporated into the formulation by adding them to the distributed transverse load.

(iv) The in-plane components of displacement in the x and y directions and the transverse component of displacement in the z direction are small in comparison with the plate thickness.

2.3. Displacement field

Ignoring the in-plane deformations, the displacement field components for flexural deformations are:

$$u(x,y,z) = z\alpha_x + z^3\beta_x = z(\alpha_x + z^2\beta_x), \tag{1a}$$
$$v(x,y,z) = z\alpha_y + z^3\beta_y = z(\alpha_y + z^2\beta_y), \tag{1b}$$
$$w(x,y,z) = w(x,y,z=0) = w_0(x,y), \tag{1c}$$

where, $u(x,y,z)$ is the displacement field component in the x coordinate direction, $v(x,y,z)$ is the displacement field component in the y coordinate direction, $w_0(x,y)$ is the transverse (z) component of the displacement of the middle surface ($z = 0$), α_x, α_y, β_x, β_y are the unknown displacement variables $\alpha_x(x,y)$ and $\alpha_y(x,y)$ are the rotations of the normals to the middle surface of the plate about the y and x axes respectively; while $\beta_x(x,y)$ and $\beta_y(x,y)$ are called the displacement warping functions.

Five unknown functions, namely $\alpha_x(x,y)$, $\alpha_y(x,y)$, $\beta_x(x,y)$, $\beta_y(x,y)$ and $w_0(x,y)$ are used to describe the displacement field components in the Reddy plate theory (RdPT). Two of these unknown functions ($\alpha_x(x,y)$, $\alpha_y(x,y)$) are encountered in the Mindlin plate theory (MdPT) which is a first order shear deformation plate theory (FSDPT). The displacement warping functions $\beta_x(x,y)$, $\beta_y(x,y)$ are derivable/expressible in terms of $\alpha_x(x,y)$ and $\alpha_y(x,y)$ by the imposition of the transverse shear stress free boundary conditions at the top and bottom faces of the plate.

2.4. Strain field (Kinematics)

Assuming infinitesimally small displacements, the strain fields are obtained from the strain-displacement relations of the small displacement linear elasticity as follows:

$$\varepsilon_{xx} = \frac{\partial u}{\partial x}, \quad \varepsilon_{yy} = \frac{\partial v}{\partial y}, \quad \varepsilon_{zz} = \frac{\partial w}{\partial z},$$
$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}, \quad \gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}, \quad \gamma_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}, \tag{2}$$

where ε_{xx}, ε_{yy} and ε_{zz} are the normal strains in the x, y and z Cartesian coordinate directions respectively; γ_{xy}, γ_{yz} and γ_{xz} are the shear strains.

By substitution of the displacement field components – Eq. (1) – into Eq. (2) we have:

$$\varepsilon_{xx} = \frac{\partial}{\partial x}(z\alpha_x + z^3\beta_x) = z\frac{\partial \alpha_x}{\partial x} + z^3\frac{\partial \beta_x}{\partial x}, \tag{3a}$$
$$\varepsilon_{yy} = \frac{\partial}{\partial x}(z\alpha_y + z^3\beta_y) = z\frac{\partial \alpha_y}{\partial y} + z^3\frac{\partial \beta_y}{\partial y}, \tag{3b}$$
$$\varepsilon_{zz} = \frac{\partial w_0}{\partial z}(x, y) = 0, \tag{3c}$$
$$\gamma_{xy} = \frac{\partial}{\partial x}(z\alpha_y + z^3\beta_y) + \frac{\partial}{\partial y}(z\alpha_x + z^3\beta_x), \tag{3d}$$
$$\gamma_{xy} = z\left(\frac{\partial \alpha_x}{\partial y} + \frac{\partial \alpha_y}{\partial x}\right) + z^3\left(\frac{\partial \beta_x}{\partial y} + \frac{\partial \beta_y}{\partial x}\right), \tag{3e}$$
$$\gamma_{yz} = \frac{\partial w_0}{\partial y} + \frac{\partial}{\partial z}(z\alpha_y + z^3\beta_y) = \frac{\partial w_0}{\partial y} + \alpha_y + 3z^2\beta_y, \tag{3f}$$
$$\gamma_{xz} = \frac{\partial w_0}{\partial x} + \frac{\partial}{\partial z}(z\alpha_x + z^3\beta_x) = \frac{\partial w_0}{\partial x} + \alpha_x + 3z^2\beta_x. \tag{3g}$$

2.5. Stress fields

The stress fields are determined by using the stress-strain relations of isotropic homogeneous elasticity. Thus:

$$\sigma_{xx} = \frac{E}{1-\mu^2}(\varepsilon_{xx} + \mu\varepsilon_{yy}), \tag{4a}$$
$$\sigma_{yy} = \frac{E}{1-\mu^2}(\varepsilon_{yy} + \mu\varepsilon_{xx}), \tag{4b}$$
$$\tau_{xy} = G\gamma_{xy}, \tag{4c}$$
$$\tau_{yz} = G\gamma_{yz}, \tag{4d}$$
$$\tau_{xz} = G\gamma_{xz}, \tag{4e}$$

where:

$$G = \frac{E}{2(1+\mu)}, \tag{5}$$

and E is the Young's modulus of elasticity, G is the shear modulus, μ is the Poisson's ratio.

Substituting the expressions for normal and shear strains into Eqs. (4a-4e), the stress fields are found as:

$$\sigma_{xx} = \frac{E}{1-\mu^2}\left(z\frac{\partial \alpha_x}{\partial x} + z^3\frac{\partial \beta_x}{\partial x}\right) + \frac{\mu E}{1-\mu^2}\left(z\frac{\partial \alpha_y}{\partial y} + z^3\frac{\partial \beta_y}{\partial y}\right), \tag{6a}$$
$$\sigma_{yy} = \frac{E}{1-\mu^2}\left(z\frac{\partial \alpha_y}{\partial y} + z^3\frac{\partial \beta_y}{\partial y}\right) + \frac{\mu E}{1-\mu^2}\left(z\frac{\partial \alpha_x}{\partial x} + z^3\frac{\partial \beta_x}{\partial x}\right), \tag{6b}$$

$$\tau_{yz} = G\left(\alpha_y + 3z^2\beta_y + \frac{\partial w_0}{\partial y}\right), \tag{6c}$$

$$\tau_{xz} = G\left(\alpha_x + 3z^2\beta_x + \frac{\partial w_0}{\partial x}\right), \tag{6d}$$

$$\tau_{xy} = Gz\left(\frac{\partial \alpha_x}{\partial y} + \frac{\partial \alpha_y}{\partial x}\right) + Gz^3\left(\frac{\partial \beta_x}{\partial y} + \frac{\partial \beta_y}{\partial x}\right). \tag{6e}$$

2.6. Enforcement of boundary conditions

The thick plate bending problem considered in this study is subjected to a distributed transverse load of intensity $q(x,y)$ on the top surface $(z = h/2)$ while the bottom surface $(z = -h/2)$ of the plate is free of load. The top and bottom surfaces of the plate $(z = \pm h/2)$ are free from shear stresses. The shear stress free boundary conditions at the top and bottom surfaces of the plate can be expressed as follows:

$$\tau_{yz}\left(x, y, z = \pm\frac{h}{2}\right) = 0, \tag{7a}$$

$$\tau_{xz}\left(x, y, z = \pm\frac{h}{2}\right) = 0. \tag{7b}$$

Applying the shear stress free boundary conditions on Eq. (6c) we have:

$$G\left(\alpha_y + 3z^2\beta_y + \frac{\partial w_0}{\partial y}\right)\Bigg|_{z=\pm h/2} = 0, \tag{8a}$$

$$G\left(\alpha_y + 3\left(\pm\frac{h}{2}\right)^2\beta_y + \frac{\partial w_0}{\partial y}\right) = 0. \tag{8b}$$

$G \neq 0$ hence:

$$\alpha_y + \frac{3h^2}{4}\beta_y + \frac{\partial w_0}{\partial y} = 0. \tag{8c}$$

Solving for β_y gives:

$$\beta_y = \frac{-4}{3h^2}\left(\alpha_y + \frac{\partial w_0}{\partial y}\right). \tag{8d}$$

Similarly, applying the boundary conditions – Eq. (7b) on Eq. (6d) gives:

$$\tau_{xz}\left(x, y, z = \pm\frac{h}{2}\right) = G\left(\alpha_x + 3z^2\beta_x + \frac{\partial w_0}{\partial x}\right)\Bigg|_{z=\pm h/2} = 0. \tag{9a}$$

$G \neq 0$. Hence:

$$\alpha_x + \frac{3h^2}{4}\beta_x + \frac{\partial w_0}{\partial x} = 0. \tag{9b}$$

Solving for β_x gives:

$$\beta_x = -\frac{4}{3h^2}\left(\alpha_x + \frac{\partial w_0}{\partial x}\right). \tag{9c}$$

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The relationships between β_x and α_x and w_0 and between β_y and α_y and w_0, thus reduce the number of unknown displacement parameters in the formulation to three, namely $\alpha_x(x,y)$, $\alpha_y(x,y)$ and $w_0(x,y)$.

2.7. Displacement field that satisfy shear stress free boundary conditions

The displacement field components are thus simplified and expressed in terms of the three unknown displacement parameters as follows:

$$u(x,y,z) = z\left(\alpha_x - \frac{4}{3}\left(\frac{z}{h}\right)^2\left(\frac{\partial w_0}{\partial x} + \alpha_x\right)\right), \tag{10a}$$

$$v(x,y,z) = z\left(\alpha_y - \frac{4}{3}\left(\frac{z}{h}\right)^2\left(\frac{\partial w_0}{\partial y} + \alpha_y\right)\right), \tag{10b}$$

$$w(x,y,z) = w_0(x,y). \tag{10c}$$

The displacement field components are now simplified and expressed in terms of only three unknown displacement parameters ($\alpha_x(x,y)$, $\alpha_y(x,y)$ and $w_0(x,y)$) as in the MdPT and other FSDPTs even though a third order displacement variation has been apriori assumed in the formulation.

2.8. Strain fields that satisfy shear stress free boundary conditions

The resulting strain fields that satisfy the shear stress free boundary conditions at the top and bottom plate surfaces are now found as follows:

$$\varepsilon_{xx} = z\left(\frac{\partial \alpha_x}{\partial x} - \frac{4}{3}\left(\frac{z}{h}\right)^2\left(\frac{\partial^2 w_0}{\partial x^2} + \frac{\partial \alpha_x}{\partial x}\right)\right), \tag{11a}$$

$$\varepsilon_{yy} = z\left(\frac{\partial \alpha_y}{\partial y} - \frac{4}{3}\left(\frac{z}{h}\right)^2\left(\frac{\partial^2 w_0}{\partial y^2} + \frac{\partial \alpha_y}{\partial y}\right)\right), \tag{11b}$$

$$\varepsilon_{zz} = 0, \tag{11c}$$

$$\gamma_{xy} = z\left[\left\{\frac{\partial \alpha_x}{\partial y} - \frac{4}{3}\left(\frac{z}{h}\right)^2\left(\frac{\partial^2 w_0}{\partial x \partial y} + \frac{\partial \alpha_x}{\partial y}\right)\right\} + \left\{\frac{\partial \alpha_y}{\partial x} - \frac{4}{3}\left(\frac{z}{h}\right)^2\left(\frac{\partial^2 w_0}{\partial x \partial y} + \frac{\partial \alpha_y}{\partial x}\right)\right\}\right], \tag{11d}$$

$$\gamma_{yz} = \left(1 - \frac{4z^2}{h^2}\right)\left(\alpha_y + \frac{\partial w_0}{\partial y}\right), \tag{11e}$$

$$\gamma_{xz} = \left(1 - \frac{4z^2}{h^2}\right)\left(\alpha_x + \frac{\partial w_0}{\partial x}\right). \tag{11f}$$

2.9. Stress fields that satisfy shear stress free boundary conditions

The stress fields that satisfy the shear stress free boundary conditions on $z = \pm h/2$ surfaces are:

$$\sigma_{xx} = \frac{Ez}{1-\mu^2}\left(\frac{\partial \alpha_x}{\partial x} - \frac{4}{3}\left(\frac{z}{h}\right)^2\left(\frac{\partial^2 w_0}{\partial x^2} + \frac{\partial \alpha_x}{\partial x}\right)\right) + \frac{\mu E z}{1-\mu^2}\left(\frac{\partial \alpha_y}{\partial y} - \frac{4}{3}\left(\frac{z}{h}\right)^2\left(\frac{\partial^2 w_0}{\partial y^2} + \frac{\partial \alpha_y}{\partial y}\right)\right), \tag{12a}$$

$$\sigma_{yy} = \frac{Ez}{1-\mu^2}\left(\frac{\partial \alpha_y}{\partial y} - \frac{4}{3}\left(\frac{z}{h}\right)^2\left(\frac{\partial^2 w_0}{\partial y^2} + \frac{\partial \alpha_y}{\partial y}\right)\right) + \frac{\mu E z}{1-\mu^2}\left(\frac{\partial \alpha_x}{\partial x} - \frac{4}{3}\left(\frac{z}{h}\right)^2\left(\frac{\partial^2 w_0}{\partial x^2} + \frac{\partial \alpha_x}{\partial x}\right)\right), \tag{12b}$$

$$\tau_{xy} = Gz\left(\frac{\partial \alpha_x}{\partial y} + \frac{\partial \alpha_y}{\partial x}\right) - G\frac{4}{3}\frac{z^3}{h^2}\left(2\frac{\partial^2 w_0}{\partial x \partial y} + \frac{\partial \alpha_x}{\partial y} + \frac{\partial \alpha_y}{\partial x}\right), \quad (12c)$$

$$\tau_{yz} = G\left(1 - \frac{4z^2}{h^2}\right)\left(\alpha_y + \frac{\partial w_0}{\partial y}\right), \quad (12d)$$

$$\tau_{zx} = G\left(1 - \frac{4z^2}{h^2}\right)\left(\alpha_x + \frac{\partial w_0}{\partial x}\right). \quad (12e)$$

Simplification gives the normal stress fields as:

$$\sigma_{xx} = \frac{Ez}{1-\mu^2}\left(\frac{\partial \alpha_x}{\partial x} + \mu\frac{\partial \alpha_y}{\partial y}\right) - \frac{4}{3}\frac{z^3}{h^2}\frac{E}{1-\mu^2}\left(\frac{\partial^2 w_0}{\partial x^2} + \frac{\partial \alpha_x}{\partial x} + \mu\frac{\partial^2 w_0}{\partial y^2} + \mu\frac{\partial \alpha_y}{\partial y}\right), \quad (13a)$$

$$\sigma_{yy} = \frac{Ez}{1-\mu^2}\left(\frac{\partial \alpha_y}{\partial y} + \mu\frac{\partial \alpha_x}{\partial x}\right) - \frac{4}{3}\frac{z^3}{h^2}\frac{E}{1-\mu^2}\left(\frac{\partial^2 w_0}{\partial y^2} + \frac{\partial \alpha_y}{\partial y} + \mu\frac{\partial^2 w_0}{\partial x^2} + \mu\frac{\partial \alpha_x}{\partial x}\right). \quad (13b)$$

2.10. Internal stress resultants

The internal stress resultants are given by the following integration problems over the plate thickness:

$$M_{xx} = \int_{-h/2}^{h/2} \sigma_{xx} z \, dz, \quad (14a)$$

$$M_{yy} = \int_{-h/2}^{h/2} \sigma_{yy} z \, dz, \quad (14b)$$

$$M_{xy} = \int_{-h/2}^{h/2} \tau_{xy} z \, dz, \quad (14c)$$

$$Q_x = \int_{-h/2}^{h/2} \tau_{xy} \, dz, \quad (14d)$$

$$Q_y = \int_{-h/2}^{h/2} \tau_{yz} \, dz, \quad (14e)$$

$$P_x = \int_{-h/2}^{h/2} \sigma_{xx} z^3 \, dz, \quad (14f)$$

$$P_y = \int_{-h/2}^{h/2} z^3 \sigma_{yy} \, dz, \quad (14g)$$

$$P_{xy} = \int_{-h/2}^{h/2} z^3 \tau_{xy} \, dz, \quad (14h)$$

$$R_x = \int_{-h/2}^{h/2} z^2 \tau_{xz} \, dz, \quad (14i)$$

$$R_y = \int_{-h/2}^{h/2} z^2 \tau_{yz} \, dz, \quad (14j)$$

where M_{xx}, M_{yy} are bending moments; M_{xy} is twisting moment; P_x, P_y, P_{xy}, R_x and R_y are the higher-order internal stress resultants.

By substitution of the expressions for the stress fields, and integration over the thickness of the plate, explicit expressions are obtained for the internal stress resultants. These expressions relate the internal stress resultants to the unknown displacements of the formulation.

Thus,

$$M_{xx}=\int_{-h/2}^{h/2}\left\{\frac{Ez^2}{1-\mu^2}\left(\frac{\partial\alpha_x}{\partial x}+\mu\frac{\partial\alpha_y}{\partial y}\right)-\frac{4}{3}\frac{E}{1-\mu^2}\frac{z^4}{h^2}\left(\frac{\partial^2 w_0}{\partial x^2}+\mu\frac{\partial^2 w_0}{\partial y^2}+\frac{\partial\alpha_x}{\partial x}+\mu\frac{\partial\alpha_y}{\partial y}\right)\right\}dz. \quad (15)$$

Simplifying:

$$M_{xx}=\left(\frac{\partial\alpha_x}{\partial x}+\mu\frac{\partial\alpha_y}{\partial y}\right)\int_{-h/2}^{h/2}\frac{Ez^2}{1-\mu^2}dz$$
$$-\left(\frac{\partial^2 w_0}{\partial x^2}+\mu\frac{\partial^2 w_0}{\partial y^2}+\frac{\partial\alpha_x}{\partial x}+\mu\frac{\partial\alpha_y}{\partial y}\right)\int_{-h/2}^{h/2}\frac{4}{3}\frac{E}{1-\mu^2}\frac{z^4}{h^2}dz.$$

Let:

$$D=\int_{-h/2}^{h/2}\frac{Ez^2}{1-\mu^2}\,dz=\frac{Eh^3}{12(1-\mu^2)}. \quad (16)$$

D is the modulus of flexural rigidity:

$$M_{xx}=D\left(\frac{\partial\alpha_x}{\partial x}+\mu\frac{\partial\alpha_y}{\partial y}\right)-\frac{D}{5}\left(\frac{\partial^2 w_0}{\partial x^2}+\mu\frac{\partial^2 w_0}{\partial y^2}+\frac{\partial\alpha_x}{\partial x}+\mu\frac{\partial\alpha_y}{\partial y}\right), \quad (17)$$

$$M_{xx}=D\left(\frac{\partial\alpha_x}{\partial x}+\mu\frac{\partial\alpha_y}{\partial y}\right)-\frac{D}{5}\left(\frac{\partial^2 w_0}{\partial x^2}+\mu\frac{\partial^2 w_0}{\partial y^2}\right)-\frac{D}{5}\left(\frac{\partial\alpha_x}{\partial x}+\mu\frac{\partial\alpha_y}{\partial y}\right), \quad (17a)$$

$$M_{xx}=\frac{4D}{5}\left(\frac{\partial\alpha_x}{\partial x}+\mu\frac{\partial\alpha_y}{\partial y}\right)-\frac{D}{5}\left(\frac{\partial^2 w_0}{\partial x^2}+\mu\frac{\partial^2 w_0}{\partial y^2}\right). \quad (17b)$$

Similarly:

$$M_{yy}=\int_{-h/2}^{h/2}\left\{\frac{Ez^2}{1-\mu^2}\left(\frac{\partial\alpha_y}{\partial y}+\mu\frac{\partial\alpha_x}{\partial x}\right)-\frac{4}{3}\frac{E}{1-\mu^2}\frac{z^4}{h^2}\left(\frac{\partial^2 w_0}{\partial y^2}+\mu\frac{\partial^2 w_0}{\partial x^2}+\frac{\partial\alpha_y}{\partial y}+\mu\frac{\partial\alpha_x}{\partial x}\right)\right\}dz, \quad (18a)$$

$$M_{yy}=\frac{4D}{5}\left(\frac{\partial\alpha_y}{\partial y}+\mu\frac{\partial\alpha_x}{\partial x}\right)-\frac{D}{5}\left(\frac{\partial^2 w_0}{\partial y^2}+\mu\frac{\partial^2 w_0}{\partial x^2}\right), \quad (18b)$$

$$M_{xy}=\int_{-h/2}^{h/2}Gz^2\left(\frac{\partial\alpha_x}{\partial y}+\frac{\partial\alpha_y}{\partial x}\right)-\int_{-h/2}^{h/2}\frac{4}{3h^2}Gz^4dz\left(2\frac{\partial^2 w_0}{\partial x\partial y}+\frac{\partial\alpha_x}{\partial y}+\frac{\partial\alpha_y}{\partial x}\right). \quad (19a)$$

Simplifying:

$$M_{xy}=\frac{Gh^3}{15}\left(\frac{\partial\alpha_x}{\partial y}+\frac{\partial\alpha_y}{\partial x}\right)-\frac{Gh^3}{30}\frac{\partial^2 w_0}{\partial x\partial y}, \quad (19b)$$

$$Q_x=\int_{-h/2}^{h/2}G\left(1-\frac{4z^2}{h^2}\right)\left(\alpha_x+\frac{\partial w_0}{\partial x}\right)dz=\frac{2Gh}{3}\left(\alpha_x+\frac{\partial w_0}{\partial x}\right), \quad (20)$$

$$Q_y = \int_{-h/2}^{h/2} G\left(1 - \frac{4z^2}{h^2}\right)\left(\alpha_y + \frac{\partial w_0}{\partial y}\right) dz = \frac{2Gh}{3}\left(\alpha_y + \frac{\partial w_0}{\partial y}\right), \tag{21}$$

$$P_{xx} = \int_{-h/2}^{h/2} \frac{Ez^4}{1-\mu^2}\left(\frac{\partial \alpha_x}{\partial x} + \mu\frac{\partial \alpha_y}{\partial y}\right) dz - \int_{-h/2}^{h/2} \frac{4}{3h^2}\frac{Ez^6}{1-\mu^2} dz \left(\frac{\partial^2 w_0}{\partial x^2} + \mu\frac{\partial^2 w_0}{\partial y^2} + \frac{\partial \alpha_x}{\partial x} + \mu\frac{\partial \alpha_y}{\partial y}\right), \tag{22a}$$

$$P_{xx} = \frac{4Dh^2}{35}\left(\frac{\partial \alpha_x}{\partial x} + \mu\frac{\partial \alpha_y}{\partial y}\right) - \frac{Dh^2}{28}\left(\frac{\partial^2 w_0}{\partial x^2} + \mu\frac{\partial^2 w_0}{\partial y^2}\right), \tag{22b}$$

$$P_{yy} = \int_{-h/2}^{h/2} \frac{Ez^4}{1-\mu^2}\left(\frac{\partial \alpha_y}{\partial y} + \mu\frac{\partial \alpha_x}{\partial x}\right) dz - \int_{-\frac{h}{2}}^{h/2} \frac{4}{3h^2}\frac{Ez^6}{1-\mu^2} dz \left(\frac{\partial^2 w_0}{\partial y^2} + \mu\frac{\partial^2 w_0}{\partial x^2} + \frac{\partial \alpha_y}{\partial y} + \mu\frac{\partial \alpha_x}{\partial x}\right), \tag{23a}$$

$$P_{yy} = \frac{4Dh^2}{35}\left(\frac{\partial \alpha_y}{\partial y} + \mu\frac{\partial \alpha_x}{\partial x}\right) - \frac{Dh^2}{28}\left(\frac{\partial^2 w_0}{\partial y^2} + \mu\frac{\partial^2 w_0}{\partial x^2}\right), \tag{23b}$$

$$P_{xy} = \int_{-h/2}^{h/2} Gz^4\, dz \left(\frac{\partial \alpha_x}{\partial y} + \frac{\partial \alpha_y}{\partial x}\right) - \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{4}{3h^2} Gz^6\, dz \left(2\frac{\partial^2 w_0}{\partial x \partial y} + \frac{\partial \alpha_x}{\partial y} + \frac{\partial \alpha_y}{\partial x}\right), \tag{24a}$$

$$P_{xy} = \frac{Gh^5}{30}\left(\frac{\partial \alpha_x}{\partial y} + \frac{\partial \alpha_y}{\partial x}\right) - \frac{Gh^5}{168}\frac{\partial^2 w_0}{\partial x \partial y}, \tag{24b}$$

$$R_x = \int_{-h/2}^{h/2} G\left(\alpha_x + \frac{\partial w_0}{\partial x}\right) z^2\left(1 - \frac{4z^2}{h^2}\right) dz = \frac{Gh^3}{30}\left(\alpha_x + \frac{\partial w_0}{\partial x}\right), \tag{25}$$

$$R_y = \int_{-h/2}^{h/2} G\left(\alpha_y + \frac{\partial w_0}{\partial y}\right) z^2\left(1 - \frac{4z^2}{h^2}\right) dz = \frac{Gh^3}{30}\left(\alpha_y + \frac{\partial w_0}{\partial y}\right). \tag{26}$$

3. Governing equations of equilibrium

The principle of minimum potential energy gives the total potential energy functional β_x and Π as follows:

$$\Pi = \int_{z=(-h/2)}^{z=h/2} \int_{R^2}\int \frac{1}{2}(\sigma_{xx}\varepsilon_{xx} + \sigma_{yy}\varepsilon_{yy} + \tau_{xy}\gamma_{xy} + \tau_{yz}\gamma_{yz} + \tau_{xz}\gamma_{xz})\, dxdydz - \iint_{R^2} qw(x,y)\, dxdy, \tag{27}$$

where R^2 is the two-dimensional (2D) domain of the plate ($0 \le x \le a$, $0 \le y \le b$), a and b are the in-plane dimensions of the plate, $q(x,y)$ is the distribution of transverse load over the plate domain.

For equilibrium:

$$\delta\prod = 0. \tag{28}$$

The equilibrium of the internal stress resultants or the principle of minimum total potential energy are used to obtain the differential equations of equilibrium of the plate. Using the equilibrium approach, the equations of equilibrium are found using:

$$\frac{\partial N_{xx}}{\partial x} + \frac{\partial N_{xy}}{\partial y} = 0, \tag{29a}$$

$$\frac{\partial N_{xy}}{\partial x} + \frac{\partial N_{yy}}{\partial y} = 0, \tag{29b}$$

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + \frac{4}{3h^2}\left(\frac{\partial^2 P_{xx}}{\partial x^2} + z\frac{\partial^2 P_{xy}}{\partial x \partial y} + \frac{\partial^2 P_{yy}}{\partial y^2}\right) + q(x,y) = 0, \tag{29c}$$

$$\frac{\partial M_{xx}}{\partial x} + \frac{\partial M_{yy}}{\partial y} - Q_x = 0, \tag{29d}$$

$$\frac{\partial M_{xy}}{\partial x} + \frac{\partial M_{yy}}{\partial y} - Q_y = 0, \tag{29e}$$

where N_{xx}, N_{yy}, N_{xy} are resultant in-plane forces.

The governing differential equations of equilibrium of the shear deformable thick plate are found in terms of the three displacement parameters as follows:

$$\left(\frac{\partial^3 w_0}{\partial x^3} + \frac{\partial^3 w_0}{\partial x \partial y^2}\right) - \frac{17}{4}\left(\frac{\partial^2 \alpha_x}{\partial x^2} + \frac{1-\mu}{2}\frac{\partial^2 \alpha_x}{\partial y^2} + \frac{1+\mu}{2}\frac{\partial^2 \alpha_y}{\partial x \partial y}\right) + \frac{21(1-\mu)}{h^2}\left(\alpha_x + \frac{\partial w_0}{\partial x}\right) = 0, \tag{30}$$

$$\left(\frac{\partial^3 w_0}{\partial y^3} + \frac{\partial^3 w_0}{\partial y \partial x^2}\right) - \frac{17}{4}\left(\frac{\partial^2 \alpha_y}{\partial y^2} + \frac{1-\mu}{2}\frac{\partial^2 \alpha_y}{\partial x^2} + \frac{1+\mu}{2}\frac{\partial^2 \alpha_x}{\partial x \partial y}\right) + \frac{21(1-\mu)}{h^2}\left(\alpha_y + \frac{\partial w_0}{\partial y}\right) = 0, \tag{31}$$

$$\nabla^4 w_0 - \frac{16}{5}\left(\frac{\partial^3 \alpha_x}{\partial x^3} + \frac{\partial^3 \alpha_x}{\partial x \partial y^2} + \frac{\partial^3 \alpha_y}{\partial x^2 \partial y} + \frac{\partial^3 \alpha_y}{\partial y^3}\right) - \frac{336}{5h^2}(1-\mu)\nabla^2 w_0 - \frac{336}{5h^2}(1-\mu)\left(\frac{\partial \alpha_x}{\partial x} + \frac{\partial \alpha_y}{\partial y}\right) - \frac{21q(x,y)}{D} = 0, \tag{32}$$

where:

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}, \tag{33}$$

$$\nabla^4 = \nabla^2\nabla^2 = (\nabla^2)^2 = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)^2, \tag{34}$$

$$\nabla^4 = \frac{\partial^4}{\partial x^4} + \frac{2\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4}. \tag{35}$$

Alternatively, the domain equations are:

$$\frac{\partial}{\partial x}(\nabla^2 w_0) - \frac{17}{4}\left(\frac{\partial^2 \alpha_x}{\partial x^2} + \frac{1-\mu}{2}\frac{\partial^2 \alpha_x}{\partial y^2} + \frac{1+\mu}{2}\frac{\partial^2 \alpha_y}{\partial x \partial y}\right) + \frac{21(1-\mu)}{h^2}\left(\alpha_x + \frac{\partial w_0}{\partial x}\right) = 0, \tag{36}$$

$$\frac{\partial}{\partial y}(\nabla^2 w_0) - \frac{17}{4}\left(\frac{\partial^2 \alpha_y}{\partial y^2} + \frac{1-\mu}{2}\frac{\partial^2 \alpha_y}{\partial x^2} + \frac{1+\mu}{2}\frac{\partial^2 \alpha_x}{\partial x \partial y}\right) + \frac{21(1-\mu)}{h^2}\left(\alpha_y + \frac{\partial w_0}{\partial y}\right) = 0, \tag{37}$$

$$\nabla^4 w_0(x,y) - \frac{16}{5}\left[\frac{\partial}{\partial x}(\nabla^2 \alpha_x) + \frac{\partial}{\partial y}(\nabla^2 \alpha_y)\right] - \frac{336}{5h^2}(1-\mu)\nabla^2 w_0 - \frac{336}{5h^2}(1-\mu)\left(\frac{\partial \alpha_x}{\partial x} + \frac{\partial \alpha_y}{\partial y}\right) - \frac{21q(x,y)}{D} = 0. \tag{38}$$

3.1. Alternative exact method for the study by Srivinas and Rao [29]

Srivinas and Rao [29] presented an alternative method for formulating and solving the candidate problem of this study. They presented a unified exact analysis for the static flexural solutions of simply supported thick plates and laminated plates based on a three-dimensional linear, small deformation theory of elasticity. Their work considered orthotropic plates, but isotropic solutions were also presented as special cases of the orthotropic material. They obtained formally exact solutions which were presented as simple infinite series for stresses and displacements for static flexural solutions.

3.2. Advantages and limitations of the third order shear deformable plate formulation

These formulated third order shear deformable plate equations avoid some issues associated with the use of lower order and simpler Kirchoff plate theory (KPT) and first order shear deformable plate theories (FSDPTs) of Timoshenko and Mindlin.

In particular, the advantages are:

1) The formulation allows the determination of a more accurate stress analysis without the use of shear correction factors of the FSDPTs of Timoshenko and Mindlin.

2) The resulting displacement field for the present theory determines a quadratic variation of shear strains and shear stresses across the thickness, with the shear strains and shear stresses vanishing at the top and bottom surfaces of the plate and attaining the maximum values at the middle plane. This leads to the satisfaction of the shear stress-free boundary conditions at the top and bottom plate surfaces $z = \pm h/2$, and the obviation of the need for shear correction factors.

3) The governing equations formulated take due consideration of transverse shear deformation in deriving the equations of equilibrium, and hence make the resulting formulation applicable to moderately thick and thick plates where transverse shear deformation is vital to their behaviour.

The disadvantages are:

1) The formulation assumes that the plate material is homogenous and hence the equations would not apply to non-homogenous plates.

2) The assumption of isotropic plate material also restricts the applicability to isotropic plates.

3) The formulation applies to linear, small displacement static flexure problems and is inapplicable to large displacement, nonlinear flexural problems.

4) The formulation contains three unknown displacements, which increase the difficulty in obtaining solutions to the governing equations as the solutions involve prior determination of the three unknown displacement parameters.

4. Application to the flexural analysis of simply supported thick plate

A rectangular thick plate with simply supported edges $x = 0$, $x = a$, $y = 0$, $y = b$ is considered. The plate has in-plane dimensions $a \times b$ and thickness h, and is subjected to a distributed transverse load of intensity $q(x,y)$ over the domain $0 \le x \le a$, $0 \le y \le b$.

The boundary conditions along the simply supported edges are given by along $x = 0$:

$$w(x=0,y) = 0, \tag{39a}$$

$$P_{xx}(x=0,y)=0, \tag{39b}$$
$$M_{xx}(x=0,y)=0, \tag{39c}$$
$$\alpha_y(x=0,y)=0. \tag{39d}$$

Similarly, along $x=a$:

$$w(x=a,y)=0, \tag{40a}$$
$$P_{xx}(x=a,y)=0, \tag{40b}$$
$$M_{xx}(x=a,y)=0, \tag{40c}$$
$$\alpha_y(x=a,y)=0. \tag{40d}$$

Along $y=0$:

$$w(x,y=0)=0, \tag{41a}$$
$$P_{yy}(x,y=0)=0, \tag{41b}$$
$$M_{yy}(x,y=0)=0, \tag{41c}$$
$$\alpha_x(x,y=0)=0. \tag{41d}$$

Along $y=b$:

$$w(x,y=b)=0, \tag{42a}$$
$$P_{yy}(x,y=b)=0, \tag{42b}$$
$$M_{yy}(x,y=b)=0, \tag{42c}$$
$$\alpha_x(x,y=b)=0. \tag{42d}$$

4.1. Navier’s double trigonometric series method of solution

Using Navier’s double trigonometric series method and considering the displacement shape functions that satisfy apriori the boundary conditions along the four edges, $w(x,y)$, $\alpha_x(x,y)$ and $\alpha_y(x,y)$ are expressed in terms of double series of infinite terms as:

$$w(x,y)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}A_{mn}\sin\lambda_m x\sin\gamma_n y, \tag{43}$$
$$\alpha_x(x,y)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}B_{mn}\cos\lambda_m x\sin\gamma_n y, \tag{44}$$
$$\alpha_y(x,y)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}C_{mn}\sin\lambda_m x\cos\gamma_n y, \tag{45}$$

where A_{mn}, B_{mn} and C_{mn} are the unknown parameters of $w(x,y)$, $\alpha_x(x,y)$, and $\alpha_y(x,y)$ respectively, and λ_m and γ_n are defined in terms of m, n, a and b as:

$$\lambda_m=\frac{m\pi}{a}, \tag{46}$$
$$\gamma_n=\frac{n\pi}{b}. \tag{47}$$

The distributed transverse load $q(x,y)$ is represented as the double Fourier sine series expansion:

$$q(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} q_{mn}\sin\lambda_m x\sin\gamma_n y, \tag{48}$$

where q_{mn} are the Fourier series coefficients of the load.

Then by Fourier series theory:

$$q_{mn} = \frac{4}{ab}\int_0^a\int_0^b q(x,y)\sin\lambda_m x\sin\gamma_n y\, dxdy. \tag{49}$$

Substitution of Eqs. (43-45) and Eq. (49) into the governing PDEs and simplification gives the following system of equations:

$$\begin{aligned}&\left(-\lambda_m(\lambda_m^2+\gamma_n^2) + \frac{21(1-\mu)}{h^2}\lambda_m\right)A_{mn} + \left(\frac{17}{4}\left(\lambda_m^2 + \left(\frac{1-\mu}{2}\right)\gamma_n^2\right) + \frac{21(1-\mu)}{h^2}\right)B_{mn}\\ &\quad + \frac{17}{8}(1+\mu)\lambda_m\gamma_n C_{mn} = 0,\end{aligned} \tag{50}$$

$$\begin{aligned}&\left(\gamma_n\left(\frac{21(1-\mu)}{h^2} - (\lambda_m^2+\gamma_n^2)\right)\right)A_{mn} + \frac{17}{8}(1+\mu)\lambda_m\gamma_n B_{mn}\\ &\quad + \left(\frac{17}{4}\left(\gamma_n^2 + \frac{1-\mu}{2}\lambda_m^2\right) + \frac{21(1-\mu)}{h^2}\right)C_{mn} = 0,\end{aligned} \tag{51}$$

$$\begin{aligned}&\left((\lambda_m^2+\gamma_n^2)^2 - \frac{336(1-\mu)}{5h^2}(\lambda_m^2+\gamma_n^2)\right)A_{mn} + \left(\frac{336(1-\mu)}{5h^2}\lambda_m - \frac{16}{5}\lambda_m(\lambda_m^2+\gamma_n^2)\right)B_{mn}\\ &\quad + \left(\frac{336(1-\mu)}{5h^2}\gamma_n - \frac{16}{5}\gamma_n(\lambda_m^2+\gamma_n^2)\right)C_{mn} = \frac{21}{D}q_{mn}.\end{aligned} \tag{52}$$

4.2. Matrix representation of the algebraic problem

The resulting algebraic problem is represented in matrix format as follows:

$$\begin{pmatrix} a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{pmatrix}\begin{pmatrix}A_{mn}\\ B_{mn}\\ C_{mn}\end{pmatrix} = \begin{pmatrix}0\\ 0\\ Q_{mn}\end{pmatrix}, \tag{53}$$

wherein:

$$\begin{aligned}
a_{11} &= \frac{21(1-\mu)}{h^2}\lambda_m - \lambda_m(\lambda_m^2+\gamma_n^2),\\
a_{12} &= \frac{17}{4}\left(\lambda_m^2 + \frac{1-\mu}{2}\gamma_n^2\right) + \frac{21(1-\mu)}{h^2},\\
a_{13} &= \frac{17}{8}(1+\mu)\lambda_m\gamma_n,\\
a_{21} &= \gamma_n\left(\frac{21(1-\mu)}{h^2} - (\lambda_m^2+\gamma_n^2)\right),\\
a_{22} &= \frac{17}{8}(1+\mu)\lambda_m\gamma_n = a_{13},\\
a_{23} &= \frac{17}{4}\left(\gamma_n^2 + \frac{1-\mu}{2}\lambda_m^2\right) + \frac{21(1-\mu)}{h^2},\\
a_{31} &= (\lambda_m^2+\gamma_n^2)^2 + \frac{336(1-\mu)}{5h^2}(\lambda_m^2+\gamma_n^2),\\
\alpha_{32} &= \frac{336(1-\mu)}{5h^2}\lambda_m^2 - \frac{16}{5}\lambda_m(\lambda_m^2+\gamma_n^2),\\
\alpha_{33} &= \frac{336(1-\mu)}{5h^2}\gamma_n - \frac{16}{5}\gamma_n(\lambda_m^2+\gamma_n^2),\\
Q_{mn} &= \frac{21q_{mn}}{D}.
\end{aligned} \tag{54}$$

Case 1: Sinusoidal distribution of transverse load:

$$q(x,y) = q_0 \sin\frac{\pi x}{a}\sin\frac{\pi y}{b}. \tag{55}$$

Then:

$$q_{mn} = \frac{4}{ab}\int_0^a\int_0^b q_0 \sin\frac{\pi x}{a}\sin\frac{\pi y}{b}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\,dxdy, \tag{56}$$

$$q_{mn} = \begin{cases} q_0, & m=n=1,\\ 0, & m>1, \quad n>1. \end{cases} \tag{57}$$

Case 2: Uniformly distributed transverse load q_0 on the surface $z=-h/2$ acting in the z direction:

$$q(x,y) = q_0, \quad 0 \le x \le a, \quad 0 \le y \le b, \tag{58}$$

$$q_{\max} = \frac{4}{ab}\int_0^a\int_0^b q_0 \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\,dxdy = \frac{16q_0}{mn\pi^2}, \quad m = 1,3,5,7,9\ldots, \quad n = 1,3,5,7,9\ldots, \tag{59a}$$

$$q_{mn} = 0, \quad m = 2,4,6,8,10\ldots, \quad n = 2,4,6,8,10\ldots. \tag{59b}$$

Case 3: Linearly varying transverse load:

$$q(x,y) = \frac{q_0 x}{a}. \tag{60}$$

Then:

$$q_{mn} = \frac{4}{ab}\int_0^a\int_0^b \frac{q_0 x}{a}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\,dxdy, \tag{61a}$$

$$q_{mn} = \frac{8q_0}{mn\pi^2}\cos m\pi. \tag{61b}$$

Case 4: Transverse point load P_0 acting at $x = \bar{x}$, $y = \bar{y}$, then:

$$q(x,y) = P_0\delta(x = \bar{x}, y = \bar{y}), \tag{62}$$

$$q_{mn} = \frac{4}{ab}\int_0^a\int_0^b P_0\delta(x = \bar{x}, y = \bar{y})\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\,dxdy, \tag{63}$$

$$q_{mn} = \frac{4P_0}{ab}\sin\frac{m\pi\bar{x}}{a}\sin\frac{n\pi\bar{y}}{b}. \tag{64}$$

For center point loads, $\bar{x} = \frac{a}{2}$, $\bar{y} = \frac{b}{2}$:

$$q_{mn} = \frac{4P_0}{ab}\sin\frac{m\pi}{2}\sin\frac{n\pi}{2}. \tag{65}$$

5. Results and discussion

For each loading case considered, non-dimensional transverse displacement $\bar{w}$ and the normal stresses $(\bar{\sigma}_{xx}, \bar{\sigma}_{yy})$ and shear stresses $(\bar{\tau}_{xy}, \bar{\tau}_{zx}, \bar{\tau}_{yz})$ are calculated at defined points on the plate as follows:

$$\bar{w} = \frac{100Ew}{qhs}, \quad \bar{\sigma}_{xx} = \frac{\sigma_{xx}}{qs^2}, \quad \bar{\sigma}_{yy} = \frac{\sigma_{yy}}{qs^2}, \quad \tau_{xy} = \frac{\tau_{xy}}{qs^2}, \quad \bar{\tau}_{zx} = \frac{\tau_{zx}}{qs}, \quad \tau_{yz} = \frac{\tau_{yz}}{qs}, \tag{66}$$

where $s = a/h$.

The results for the dimensionless displacements and stresses are shown in Tables 1-4 for the four cases of load distributions considered in the study. In the tables, $\bar{\tau}_{zx}$ and $\bar{\tau}_{yz}$ were evaluated using two procedures namely constitutive relations, which is denoted by CR and equilibrium equations denoted by EE as superscripts for each of the stress notations. The percentage error in the results obtained and other results from literature sources are calculated by comparison with the exact solution as follows:

$$\%\text{ error} = \frac{\text{value of result} - \text{exact result}}{\text{exact result}} \times 100\ \%. \tag{67}$$

The values enclosed in brackets in Tables 1-4 are the % errors for the displacements and stresses relative to the exact solutions computed by Srivinas and Rao [29] as indicated in the corresponding table.

Table 1 presents the results for dimensionless displacement $\bar{w}$ and stresses for the case of sinusoidal load distribution on a square thick plate for $h/a = 0.10$ and for $h/a = 0.25$. Table 1 shows that the present results for $\bar{w}$ at the center (for $h/a = 0.10$) is 0.62 % greater than the exact result obtained by Srivinas and Rao [29]. Table 1 further shows that Krishna Murty's HSDPT solutions gave a better result for $\bar{w}$ at the center as the result coincides with the exact result. However, the present result gave better result than the Kirchhoff CPT result which showed a percentage error of –4.76 %.

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Table 1. Dimensionless deflection $\bar{w}$ at $(x = a/2, y = b/2, z = 0)$, Dimensionless in-plane normal stress $\bar{\sigma}_{xx}$ at $(x = a/2, y = b/2, z = h/2)$, $\bar{\sigma}_{yy}$ at $(x = a/2, y = b/2, z = h/2)$ in-plane shear stress $\bar{\tau}_{xy}$ at $(x = 0, y = 0, z = h/2)$ and transverse shear stress $\bar{\tau}_{zx}$ at $(x = 0,\ y = b_x,\ z = 0)$ in square thick plate subjected to single sine load

$\frac{a}{b}$	$\frac{h}{a}$	Theory	$\bar{w}$	$\bar{\sigma}_{xx}$	$\bar{\sigma}_{yy}$	$\bar{\tau}_{xy}$	$\bar{\tau}_{zx}^{CR}$	$\bar{\tau}_{zx}^{EE}$
1	0.1	Present work	2.960 (0.62 %)	0.289 (0.00 %)	0.2890 (0.00 %)	0.1070	0.2380 (0.00 %)	0.228
		Krishna Murty [28] HSDT	2.942 (0.00%)	0.290 (0.34 %)	0.290 (0.34 %)	0.107	0.238 (0.00 %)	0.228
		Kirchhoff [6] (CPT)	2.802 (–4.76 %)	0.287 (–1.50%)	0.287 (–1.50 %)	0.106	–	0.238
		(Exact) Srivinas et al [29]	2.942	0.289	0.289	–	0.238	–
	0.25	Present Study	3.7810 (3.39%)	0.2090 (2.45 %)	0.2090 (2.45%)	0.1120	0.2370 (0.38 %)	–
		Sayyad and Ghugal [31]	3.7480 (2.32 %)	0.2130 (4.41%)	0.2130 (4.41 %)	0.1140	0.2380 (0.80 %)	–
		Kirchhoff [6]	2.8030 (–23.48 %)	0.1970 (–3.43 %)	0.1970 (–3.43 %)	0.1060	–	–
		(Exact) Srivinas and Rao [29]	3.6630	0.2040	0.2040	–	0.2361	–

Table 2. Non-dimensionless transverse deflection $\bar{w}$ at $(x = a/2, y = b/2, z = 0)$, Dimensionless in-plane stresses $\bar{\sigma}_{xx}$ at $(x = a/2, y = b/2, z = h/2)$, $\bar{\sigma}_{yy}$ at $(x = a/2, y = b/2, z = h/2)$ in-plane shear stress $\bar{\tau}_{xy}$ at $(x = 0, y = 0, z = h/2)$ and transverse shear stress $\bar{\tau}_{zx}$ at $(x = 0, y = b/2, z = 0)$ in rectangular thick plate subjected to uniformly distributed loading

$\frac{a}{b}$	$\frac{h}{a}$	Theory / Method / Procedure	$\bar{w}$	$\bar{\sigma}_{xx}$	$\bar{\sigma}_{yy}$	$\bar{\tau}_{xy}$	$\bar{\tau}_{zx}^{CR}$	$\bar{\tau}_{zx}^{EE}$
0.5	0.10	Present Study	11.420 (0.39 %)	0.612 (0.00 %)	0.278 (–1.06 %)	0.280	0.679 (0.00 %)	0.6776 (–0.206 %)
		Krishna Murty [28] HSDT	11.310 (–0.57 %)	0.613 (0.163 %)	0.310 (10.32 %)	0.278	0.682 (0.441 %)	0.667 (–1.76 %)
		Kirchhoff [6] (CPT)	11.060 (–2.78 %)	0.610 (–0.32 %)	0.278 (–1.06 %)	0.277	0.686 (1.03 %)	0.6865 (1.104 %)
		(Exact) Srivinas and Rao [29]	11.375	0.612	0.281	–	0.679	–

Table 1 also reveals that the present results for $\bar{\sigma}_{xx}$ and $\bar{\sigma}_{yy}$ (at $x = a/2$, $y = b/2$, $z = h/2$) coincide with the exact result obtained by Srivinas et al [29] while Krishna Murty's results for $\bar{\sigma}_{xx}$ and $\bar{\sigma}_{yy}$ are 0.34 % in error and CPT of Kirchhoff is –1.50 % in error.

For the case of sinusoidal load distribution for $h/a = 0.25$, the present result for $\bar{w}$ at the center is 3.39 % different from the exact result given by Srivinas et al [29], while the exponential shear deformation plate theory solution by Sayyad and Ghugal [31] gave better result for $\bar{w}$ at the center as the error is 2.32 %. The error of the CPT is however –23.48 % for $\bar{w}$ at the center for $h/a = 0.25$.

The present method gave more accurate results as the error is 2.45 % compared with 4.41 % error of the ESDPT of Sayyad and Ghugal and –3.43 % error of the CPT.

In the both cases of $h/a = 0.1$ and $h/a = 0.25$, the $\bar{\tau}_{xy}$was calculated by the present method, but was not determined by the exact results of Srivinas et al [29]. In both cases, the present study gave satisfactory results for $\bar{\tau}_{zx}$.

Table 2 presents non-dimensional values of $\bar{w}$ and stresses for rectangular thick plate (with $a/b = 0.5$, $h/a = 0.10$) subjected to uniformly distributed loading. Table 2 shows that the present results for $\bar{w}$ at the center has a relative difference of 0.39 % as compared with the exact solution by Srivinas et al [29]. The present result for $\bar{w}$ is more accurate than the result by Krishna Murty's HSDPT which has an error of –0.57 % and the CPT which has an error of –2.78 %. Table 2 further shows that the present results for $\bar{\sigma}_{xx}$ is exact as it coincides with the exact solution by Srivinas et al [29], while Krishna Murty's result gave an error of 0.163% and CPT gave an error of –0.32 %.

The present result for $\bar{\sigma}_{yy}$ at the point $(x = a/2, y = b/2, z = h/2)$ gave an error of –1.06 % compared with the exact result by Srivinas et al [29]. The present result is more accurate than Krishna Murty's HSDPT result which has an error of 10.32 %. The present result for $\bar{\tau}_{zx}^{CR}$ gave an exact result compared with Krishna Murty's HSDPT result which has an error of 0.441 % and CPT with an error of 1.03 %.

Table 3. Dimensionless deflection $\bar{w}$ at $(x = a/2, y = b/2, z = 0)$, in-plane normal stress $\bar{\sigma}_{xx}$ at $(x = a/2, y = b/2, z = h/2)$, $\bar{\sigma}_{yy}$ at $(x = a/2, y = b/2, z = h/2)$ in-plane shear stress $\bar{\tau}_{xy}$ at $(x = 0, y = 0, z = h/2)$ and transverse shear stress $\bar{\tau}_{zx}$ at $(x = 0, y = b/2, z = 0)$ in square and rectangular thick isotropic plate subjected to linearly distributed loading

$\frac{a}{b}$	$\frac{h}{a}$	Theory / Method / Reference	$\bar{w}$	$\bar{\sigma}_{xx}$	$\bar{\sigma}_{yy}$	$\bar{\tau}_{xy}$	$\bar{\tau}_{zx}^{CR}$	$\bar{\tau}_{zx}^{EE}$
0.5	0.1	Present work	5.7100 (0.395 %)	0.3060 (0.00 %)	0.139 (–1.068 %)	0.1400	0.3395 (0.00 %)	0.3388 (–0.206 %)
		Krishna Murty [28] HSDT	5.6878 (0.0017 %)	0.3067 (0.228 %)	0.155 (10.32 %)	0.1390	0.341 (0.441 %)	0.3375 (–0.589 %)
		Kirchhoff [6] (CPT)	5.5300 (–2.772 %)	0.3048 (–0.39 %)	0.139 (–1.068 %)	0.1385	0.343 (1.03 %)	0.3433 (1.104 %)
		(Exact) Srivinas and Rao [29]	5.6875	0.3060	0.1405	–	0.3395	–
1	0.1	Present Study	2.3325 (0.560 %)	0.1445 (0.00 %)	0.1445 (0.00 %)	0.0995	0.2463 (0.984 %)	0.243 (–0.20 %)
		Krishna Murty [28] HSDT	2.3199 (0.017 %)	0.1453 (0.553 %)	0.1453 (0.553 %)	0.0975	0.2454 (0.615 %)	0.240 (–1.44 %)
		Kirchhoff [6]	2.2180 (–4.375 %)	0.1435 (–0.007 %)	0.1435 (–0.007 %)	0.0975	–	0.247 (1.64 %)
		(Exact) Srivinas and Rao [29]	2.3195	0.1445	0.1445	–	0.2439	–

Table 3 presents the results for $\bar{w}$ and stresses for square and rectangular thick plates subjected to linearly distributed loading. For rectangular thick plate $h/a = 0.1$, $a/b = 0.5$, the present study gave $\bar{w}$ that is 0.395 % in error compared with the exact result by Srivinas et al [29]. Krishna Murty's HSDPT result with an error of 0.0017 % gave better accuracy while the present study is more accurate than the CPT which has –2.772 % error. Table 3 also shows that the present method gave accurate results for $\bar{\sigma}_{xx}$ and $\bar{\sigma}_{yy}$ than the Krishna Murty's HSDPT and Kirchhoff's CPT results. For $\bar{\sigma}_{xx}$ the present result is identical with the exact result while Krishna Murty had an error of 0.228 % and CPT an error of –0.39 %. The present method gave exact results for $\bar{\tau}_{zx}^{CR}$ while Krishna Murty's HSDPT result had an error of 0.441 % and CPT had an error of 1.03 %.

For square thick plate presented in Table 3, the present study gave an error of 0.56 % for $\bar{w}$ at the center compared with 0.017 % error of the Krishna Murty HSDPT and –4.375 % error of the CPT. Table 3 also shows that the present results for $\bar{\sigma}_{xx}$ and $\bar{\sigma}_{yy}$ are exact compared with the Srivinas et al [29] results; and are more accurate than Krishna Murty's and CPT results.

Table 4 presents the results for $\bar{w}$ and stresses for thick square plates for $h/a = 0.10$ subjected to center point load. Table 4 shows that the present study gave an error of 2.701 % for $\bar{w}$ at the center which is more accurate than the CPT results with an error of –10.173 %. Moreover, Table 4 shows that Krishna Murty's HSDPT gave more accurate result for $\bar{w}$ at the center as the error is – 0.063 %. However, for $\bar{\sigma}_{xx}$, the present results gave an error of –2.481 % showing better accuracy than the Krishna Murty's result (which has –6.609 % error) and CPT results (which has –27.03 % error).

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Table 4. Dimensionless transverse deflection $\bar{w}$ at $(x = a/2, y = b/2, z = 0)$, normal stress $\bar{\sigma}_{xx}$ at $(x = a/2, y = b/2, z = h/2)$, and transverse shear stress $\bar{\tau}_{zx}$ at $(x = 0, y = b/2, z = 0)$ in square thick isotropic plate subjected to center point load

$\frac{a}{b}$	$\frac{h}{a}$	Method / Reference	$\bar{w}$	$\bar{\sigma}_{xx}$	$\bar{\tau}_{zx}^{CR}$
1.5	0.10	Present study	14.4717 (2.701 %)	2.4956 (–2.481 %)	0.9174 (24.66 %)
		Krishna Murty [28] HSDT	14.0821 (–0.063 %)	2.3902 (–6.609 %)	0.7936 (7.840 %)
		Kirchhoff [6] (CPT)	12.6575 (–10.173 %)	1.8672 (–27.03 %)	–
		(Exact) Srivinas and Rao [29]	14.0910	2.5591	0.7359

6. Conclusions

The study has presented from first principles, a third-order shear deformation plate bending formulation for thick plates.

1) The formulation is displacement based and includes the effect of transverse shear deformations thus making it applicable to thick plates.

2) The formulation is based on infinitesimally small displacement assumptions, and linear 3D elasticity kinematics are used.

3) The resulting formulation satisfies the transverse shear stress free boundary conditions at the top and bottom surfaces of the plate, and hence transverse shear stress correction factors of the Mindlin first-order shear deformation plate theories are not needed in the present formulation.

4) The resulting governing differential equations of equilibrium are a set of three coupled differential equations in terms of three unknown displacements, namely $w_0(x,y)$, $\alpha_x(x,y)$ and $\alpha_y(x,y)$.

5) The resulting formulation is solved for simply supported rectangular and square thick plates subjected to four cases of loading distributions:

– sinusoidal loading
– uniformly distributed loading
– linearly distributed loading
– point load at the plate center.

6) For simply supported boundaries, Navier's double trigonometric series method is used to construct trial solutions for $w(x,y)$, $\alpha_x(x,y)$ and $\alpha_y(x,y)$ that apriori satisfy all the boundary conditions. The trial solutions are expressed in terms of unknown displacement parameters A_{mn}, B_{mn}, C_{mn}.

7) Navier's double trigonometric series method reduces the integration problem over the 2D domain of the plate to an algebraic problem where the unknown displacement parameters are A_{mn}, B_{mn}, C_{mn} for each load case studied.

8) Unlike the Kirchhoff plate theory (KPT) the presented formulation can be used for both thin and thick plates and yields parabolic (quadratic) variation of transverse shear stresses across the plate thickness, in agreement with theory of elasticity methods.

9) The presented formulation gives exact results for σ_{xx} (at $x = a/2$, $y = b/2$, $z = h/2$) for sinusoidal loading (for $b/a = 0.1$, $a/b = 1$); uniformly distributed and linearly distributed loads (for $h/a = 0.1$ and $a/b = 0.5$).

10) The displacements and stresses obtained by the present formulations are in good agreement with the Krishna Murty's HSDPT results.

11) The present formulation gave accurate results for transverse deflection because of the inclusion of transverse normal strain effects in the formulation.

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Data availability

The datasets generated during and/or analyzed during the current study are available from the corresponding author on reasonable request.

Conflict of interest

The authors declare that they have no conflict of interest.

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