Dynamic stiffness method for free vibration analysis of thin functionally graded rectangular plates

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Abstract. In this present work, the dynamic stiffness method (DSM) is used to analyze the free vibration of a thin functionally graded rectangular plate. Classical plate theory (CPT) is used to develop the dynamic stiffness matrix of a functionally graded material (FGM) plate. For free vibration analysis, the natural frequencies of the functionally graded material plate are estimated by using DSM with Wittrick-Williams algorithm for different aspect ratios and different boundary conditions. The present research compared the DSM natural frequencies results with those available in the published literature.

Keywords: dynamic stiffness method, free vibration, functionally graded material, CPT.

1. Introduction

The concept of functionally graded materials was first time introduced by Yamanoushi et.al [1] in 1980 during the advancement of thermal resistance material for aerospace engineering applications. Functionally graded materials are known as a new class of composite materials, which is a mixture of ceramics and metal constituents. The ceramic constituents give high-temperature resistance, whereas metal constituents enhance the mechanical performance and decrease the failure possibility of the structure. Leissa [2] used the Ritz method to analyze free vibration behaviour of the rectangular isotropic plate under applied twenty-one possible boundary conditions. Bercin [3] analyze free vibration and mode shape of the orthotropic plate by using finite element method. Bercin and Langley [4] continued to this work to develop the dynamic stiffness matrix for vibration analysis of plate structures. Boscolo and Banerjee [5] used DSM for analysis of free transverse vibration of the rectangular isotropic plate by using classical plate theory and first-order shear deformation theory. Chauhan et al. [6] used classical plate theory to analyze the free vibration of isotropic plate for different boundaries by using DSM Shen and Yang [7] applied CPT to investigate free vibration behavior of initially stressed elastically founded functionally graded material (FGM) plates under impetuous lateral loading. Baferani et al. [8] used Navier and Levy type solution for the free vibration analysis of functionally graded plate under different boundary conditions by using CPT. Kumar et al. [9] used CPT to formulate the DSM with Wittrick-Williams algorithm to extarct the eigen value of the FGM plates.

In this paper, we have analyzed the free vibration behavior of functionally graded material plates by using dynamic stiffness method with Wittrick-Williams algorithm to extract the natural frequencies under different boundary conditions.

2. Governing differential equation of the functionally graded material plate

Fig. 1. shows a rectangular functionally graded plate of length a, width b and thickness h, where material properties vary along with the thickness as a power-law distribution [9] as given by Eq. (1):

$$V_c(z) = \left(\frac{z}{h} + \frac{1}{2}\right)^k, \quad V_m(z) = 1 - V_c(z), \quad (-0.5h \leq z \leq 0.5h), \tag{1}$$

where V_c and V_m denotes the volume fractions of ceramics and metal constituents, k represent the power-law index that takes a positive real number in Eq. (1).

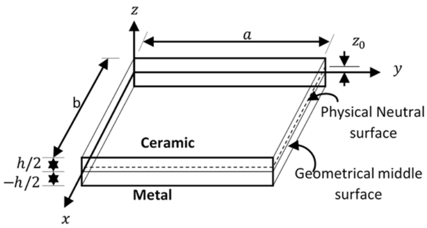

Fig. 1. Material geometry and coordinates system of the functionally graded plate

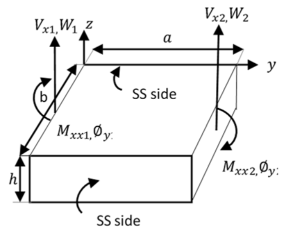

Fig. 2. Boundary conditions for displacements and forces for a plate element

The displacement components of thin rectangular functionally graded plate $u_o(x,y,z)$, $v_o(x,y,z)$ and $w_o(x,y,z)$ by using classical plate theory are given by Eq. (2):

$$u_o(x,y,z) = u'(x,y) - (z - z_0)\frac{\partial w'}{\partial x}, \quad v_o(x,y,z) = v'(x,y) - (z - z_0)\frac{\partial w'}{\partial y}, \quad w_o(x,y,z) = w'(x,y), \tag{2}$$

where $u'(x,y)$, $v'(x,y)$ and $w'(x,y)$ are the mid-plate (i.e, $z = 0$) displacement components.

Fig. 1. shows that the material properties are nonhomogeneous in the transverse direction, due to this the middle surface of the geometry has in-plane displacement, which cannot be neglected. Therefore, the middle surface of FGM plate geometry does not concur with the neutral surface. In this condition, the neutral surface must be changed to $z_n = z - z_0$, where z_0 is the distance between mid-surface to the neutral surface of the plate as shown in Fig. 1.

Hamilton's principle is used to drive the fourth-order differential equation for transverse deflection of a thin rectangular functionally graded plate under free vibration condition and is given by Eq. (3):

$$D_{eff}\left(\frac{\partial^4 w'}{\partial x^4} + 2\frac{\partial^4 w'}{\partial x^2 \partial y^2} + \frac{\partial^4 w'}{\partial y^4}\right) + \rho h \frac{\partial^4 w'}{\partial t^4} = 0. \tag{3}$$

The boundary conditions for Levy-type solution in Fig. 2., are given as:

$$V_x: -D_{eff}\left(\frac{\partial^3 w'}{\partial x^3} + (2 - \upsilon)\frac{\partial^3 w'}{\partial x \partial y^2}\right)\delta w', \quad M_{xx}: -D_{eff}\left(\frac{\partial^2 w'}{\partial x^2} + \upsilon \frac{\partial^2 w'}{\partial y^2}\right)\delta \emptyset_y, \tag{4}$$

where $D_{eff} = Eh^3/12(1 - \upsilon^2)$ is the effective bending stiffness, h plate thickness, E Young's Modulus of Elasticity, υ Poisson's ratio of the given material, V_x, M_{xx}, and $\emptyset_y$ are the shear force, bending moment and rotation of the bending plate.

3. Formulation of dynamic stiffness

A levy type solution of Eq. (3) which satisfies the boundary condition of Eq. (4) can be expressed in the following form [8]:

$$w'(x,y,t) = \sum_{m=1}^{\infty} W_m(x) e^{i\omega t} \sin(\propto_m y), \quad \propto_m = \frac{m\pi}{L}, \quad (m = 1,2,\ldots,\infty), \tag{5}$$

where ω is unknow natural frequency. By putting Eq. (5) into Eq. (3) we get Eq. (6):

$$\frac{d^4W_m}{dx^4} - 2 \propto_m^2 \frac{d^2W_m}{dx^2} + \left(\propto_m^4 - \frac{\rho h\omega^2}{D_{eff}}\right) W_m = 0, \quad (m = 1,2, \ldots, \infty). \tag{6}$$

The two possible solutions of the ordinary differential Eq. (6) are obtained, depending on the nature of all roots. Here we show only one possible solution:

Case 1: $\propto_m^2 \geq \omega \sqrt{\frac{I_0}{D_{eff}}} \Rightarrow$ all roots are real $(\propto_{1m}, -\propto_{1m}, \propto_{2m}, -\propto_{2m})$:

$$\propto_{1m} = \sqrt{\propto_m^2 + \omega \sqrt{\frac{I_0}{D_{eff}}}}, \quad \propto_{2m} = \sqrt{\propto_m^2 - \omega \sqrt{\frac{I_0}{D_{eff}}}}. \tag{7}$$

The solution is:

$$W_m(x) = A_m \cosh(\propto_{1m} x) + B_m \sinh(\propto_{1m} x) + C_m \cosh(\propto_{2m} x) + D_m \sinh(\propto_{2m} x). \tag{8}$$

The displacement w' in Eq. (8) and Eq. (5), shear force V_x, rotation $\emptyset_y$ and the bending moment M_{xx} can be expressed in the following form using Eq. (4) as shown below:

$$\phi_{ym}(x,y) = \phi_{ym}(x) \sin(\propto_m y), \tag{9}$$
$$V_{xm}(x,y) = V_{xm}(x) \sin(\propto_m y), \tag{10}$$
$$M_{xxm}(x,y) = M_{xxm}(x) \sin(\propto_m y). \tag{11}$$

The displacements boundary conditions for the plate are:

$$\begin{aligned} x = 0, \quad & W_m = W_1, \quad \phi_{ym} = \phi_{y1}, \\ x = b, \quad & W_m = W_2, \quad \phi_{ym} = \phi_{y2}, \end{aligned} \tag{12}$$

similarly, the forces boundary conditions are:

$$\begin{aligned} x = 0, \quad & V_{xm} = -V_1, \quad M_{xxm} = -M_1, \\ x = b, \quad & V_{xm} = -V_2, \quad M_{xxm} = M_2. \end{aligned} \tag{13}$$

The displacement boundary conditions are applied, i.e., putting Eq. (12) into Eqs. (8) and (9), the following matrix relationship is obtained:

$$\begin{bmatrix} W_1 \\ \phi_{y1} \\ W_2 \\ \phi_{y1} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & -\propto_{1m} & 0 & -\propto_{2m} \\ C_{h1} & S_{h1} & C_2 & S_2 \\ -\propto_{1m} S_{h1} & -\propto_{1m} C_{h1} & -\propto_{1m} S_{h2} & -\propto_{1m} C_{h1} \end{bmatrix} \begin{bmatrix} A_m \\ B_m \\ C_m \\ D_m \end{bmatrix}, \tag{14}$$
$$\delta = AC, \tag{15}$$

where $C_{h1} = \cosh(\propto_{im} b)$, $S_{h1} = \sinh(\propto_{im} b)$, $C_i = \cos(\propto_{im} b)$, $S_i = \sin(\propto_{im} b)$, $(i = 1, 2)$.

The force boundary conditions are applied, i.e., putting Eq. (13) into Eqs. (10) and (11), the following matrix relationship is obtained:

$$\begin{bmatrix} V_1 \\ M_1 \\ V_2 \\ M_2 \end{bmatrix} = \begin{bmatrix} 0 & R_1 & 0 & R_2 \\ L_1 & 0 & L_1 & 0 \\ -R_1 S_{h1} & -R_1 C_{h1} & -R_1 S_2 & -R_1 C_2 \\ -L_1 C_{h1} & -L_1 S_{h1} & -L_2 C_{h1} & -L_2 S_2 \end{bmatrix} \begin{bmatrix} A_m \\ B_m \\ C_m \\ D_m \end{bmatrix}, \tag{16}$$
$$F = RC, \tag{17}$$

where $R_i = D_{eff}(\propto_{im}{}^3 - \propto^2 \propto_{im} (2-\nu))$, $L_i = D_{eff}(\propto_{im}{}^2 - \propto^2 \nu)$ with $i = 1, 2$. Using Eqs. (15) and (17), the dynamic stiffness matrix K for functionally graded (FG) plate can be formulated by eliminating the constant vector C to get Eq. (18):

$$F = K\delta, \tag{18}$$

where:

$$K = RA^{-1}. \tag{19}$$

By using Eq. (19), the generalized dynamic stiffness matrix (K) as given by Eq. (20):

$$\boldsymbol{K} = \begin{bmatrix} s_{vv} & s_{vm} & f_{vv} & f_{vm} \\ & s_{mm} & -f_{vm} & f_{mm} \\ & Sym & s_{vv} & -s_{vm} \\ & & & s_{mm} \end{bmatrix}, \tag{20}$$

where six variable terms s_{vv}, s_{vm}, s_{mm}, f_{vv}, f_{vm}, f_{mm} can be expressed in the following form [9].

Table 1. Non-dimensional natural frequencies ($\varpi = \omega a^2\sqrt{\rho_c h/D_c}$) for Functionally graded square plates with S-S-S-S and S-F-S-F boundary conditions using DSM method

S-S-S-S							
mn	$k = 0$	$k = 0.2$	$k = 0.5$	$k = 1$	$k = 2$	$k = 5$	$k = 10$
1 1	19.7392	18.3137	16.7142	15.0610	13.6930	12.9831	12.5724
1 2	49.3480	45.7843	41.7855	37.6525	34.2326	32.4578	31.4311
2 1	49.3480	45.7843	41.7855	37.6525	34.2326	32.4578	31.4311
2 2	78.9568	73.2550	66.8568	60.2440	54.7722	51.9324	50.2898
1 3	98.6960	91.5687	83.5710	75.3050	68.4652	64.9156	62.8623
3 1	98.6960	91.5687	83.5710	75.3050	68.4652	64.9156	62.8623
2 3	128.3048	119.0393	108.6423	97.8965	89.0048	84.3902	81.7210
3 2	128.3048	119.0393	108.6423	97.8965	89.0048	84.3902	81.7210
4 1	167.7832	155.6668	142.0708	128.0186	116.3909	110.3565	106.8660
S-F-S-F							
1 1	9.6313	8.9358	8.1553	7.34874	6.6812	6.33487	6.13450
2 1	16.1347	14.9696	13.6621	12.3108	11.1926	10.6123	10.2767
1 3	36.7256	34.0735	31.0975	28.0216	25.4765	24.1556	23.3916
2 1	38.9449	36.1325	32.9767	29.7149	27.0160	25.6153	24.8051
2 2	46.7381	43.3629	39.5756	35.6611	32.4221	30.7412	29.7688
2 3	70.7401	65.6316	59.8993	53.9746	49.0722	46.5280	45.0564
1 4	75.2833	69.8468	63.7463	57.4412	52.2239	49.5163	47.9501
3 1	87.9866	81.6327	74.5029	67.1338	61.0361	57.8717	56.0412
3 2	96.0405	89.1049	81.3224	73.2788	66.6231	63.1689	61.1709

4. Numerical results

The dynamic stiffness matrix is used to obtain natural frequencies of the functionally graded plate by applying the Wittrick-Williams algorithm [5]. The above procedure is used to formulate DSM and this procedure has been implemented in MATLAB program to compute the natural frequencies of the FGM plate for different boundary conditions with different power-law index (k) values as shown in Tables 1-3, where ρ_c and D_c are denotes the density, bending stiffness of the ceramic material. The letter m denotes the number of half-sine wave in x direction, whereas n represents the nth lowest frequency of a given value of m.

Table 2. Comparison of Non-dimensional natural frequencies ($\varpi = \omega a^2\sqrt{\rho_c h/D_c}$) with results reported in the available published literature of the functionally graded plate

			S-S-S-S				S-C-S-C			
Mode	$\frac{a}{b}$	Source	$k=0$	$k=0.5$	$k=1$	$k=2$	$k=0$	$k=0.5$	$k=1$	$k=2$
1	1	DSM	19.7392	16.7142	15.0610	13.6930	28.9508	24.5141	22.0894	20.0831
		Ref [11]	19.7398	16.7141	15.0609	13.6930	28.9468	24.5122	22.0840	20.0809
		Ref [10]	19.7381	16.7127	15.0595	13.6917	28.9485	24.5122	22.0874	20.0809
		Ref [8]	19.7281	16.6879	15.0357	13.6808	28.9478	24.4867	22.0743	20.0586
		Ref [2]	19.7392	–	–	–	28.9508	–	–	–
	0.5	DSM	12.3370	10.4463	9.4131	8.5581	13.6857	11.5884	10.4422	9.4937
		Ref [8]	12.3259	10.4424	9.3849	8.5257	13.6808	11.5659	10.4093	9.484
		Ref [2]	12.3370	–	–	–	13.6858	–	–	–
2	1	DSM	49.3480	41.7855	37.6525	34.2326	54.7430	46.3537	41.7689	37.9751
		Ref [11]	49.3487	41.7852	37.6530	34.2334	54.7395	46.3525	41.7667	37.9740
		Ref [10]	49.3486	41.7868	37.6446	34.2250	54.7328	46.3424	41.7600	37.9656
		Ref [8]	49.3468	41.7894	37.6387	34.2020	54.7232	46.3297	41.7364	37.9362
		Ref [2]	49.3480	–	–	–	54.7431	–	–	–
	0.5	DSM	19.7392	16.7142	15.061	13.6930	23.6463	20.0225	18.0421	16.4034
		Ref [8]	19.7281	16.7142	15.0610	13.6931	23.6463	19.9925	18.0098	16.3905
		Ref [2]	19.7392	–	–	–	23.6463	–	–	–
3	1	DSM	78.9568	66.8568	60.2440	54.7721	94.5852	80.0902	72.1685	65.6136
		Ref [11]	78.9559	66.8569	60.2428	54.7714	94.5854	80.0902	72.1687	65.6134
		Ref [10]	78.9307	66.8351	60.2243	54.7546	94.5552	80.0633	72.1435	65.5882
		Ref [8]	78.9125	66.8173	60.2088	54.6721	94.5430	80.0360	72.1382	65.5621
		Ref [2]	78.9568	–	–	–	94.5853	–	–	–
	0.5	DSM	32.0762	27.1605	24.4741	22.2512	38.6939	32.7641	29.5234	26.8419
		Ref [8]	32.0541	27.1303	24.4536	22.2396	38.6932	32.7480	29.5096	26.8329
		Ref [2]	32.0762	–	–	–	38.6939	–	–	–

Table 3. Comparison of Non-dimensional natural frequencies ($\varpi = \omega a^2\sqrt{\rho_c h/D_c}$ of square FGM plate with published results in Chakraverty and Pradhan [12]

S-S-S-S	$k=0$			$k=0.5$			$k=1.0$		
mn	DSM	Ref [12]	%Err	DSM	Ref [12]	%Err	DSM	Ref [12]	%Err
1 1	19.7392	19.739	0.00	16.7142	17.337	3.726	15.0610	16.424	9.049
1 2	49.3480	49.349	0.00	41.7855	43.344	3.729	37.6525	41.061	9.052
2 1	49.3480	49.349	0.001	41.7855	43.344	3.729	37.6525	41.061	9.052
2 2	78.9568	79.401	0.450	66.8568	69.738	4.309	60.2440	66.065	9.662
1 3	98.6960	100.17	1.493	83.5710	87.983	5.279	75.3050	83.349	10.681
3 1	98.6960	100.19	1.513	83.5710	87.995	5.293	75.3050	83.360	10.681
S-F-S-F									
1 1	9.6313	9.632	0.007	8.1553	8.460	3.736	7.34874	8.014	9.052
2 1	16.1347	16.135	0.001	13.6621	14.172	3.732	12.3108	13.425	9.050
1 3	36.7256	37.181	1.24	31.0975	32.656	5.011	28.0216	30.936	10.400
2 1	38.9449	38.972	0.069	32.9767	34.229	3.797	29.7149	32.427	9.127
2 2	46.7381	47.281	1.161	39.5756	41.527	4.930	35.6611	39.340	10.316
2 3	70.7401	72.053	1.855	59.8993	63.285	5.652	53.9746	59.952	11.074

From Table 1, we observed that with increase in k value, the natural frequencies decrease. This is because as the k value increase, the metal constituent in the FGM plate and the stiffness of the plate is reduced.

When we compared the natural frequency results of the FGM plates with those available in the published literature, we found that the reported natural frequencies values at $k = 0$ in Tables 2-3 are nearly same with those available in the literature [2, 11, 12]. While increasing the k value from

0.5 to 1.0, the maximum error increases 5 % to 11 % as given by Chakravarty and Pradhan [12] in Table 3. The possible reasons for these reported results are discussed below.

Chakravarty and Pradhan [12] have considered mid-plane surface geometry instead of the neutral surface for solving the effective bending stiffness (D_{eff}), which increases the percentage error. Due to this reason, we have observed that error is smaller for $k = 0$ and higher for $k = 1$.

5. Conclusions

The impetus of the present work is to formulate the dynamic stiffness matrix to estimate the natural frequencies of a thin rectangular functionally graded plate, where two different sides of the plate are simply supported. Classical plate theory is used to develop the dynamic stiffness matrix of a functionally graded material plate whereas the transcendental nature of dynamic stiffness matrix is solved by using Wittrick-Williams algorithm and this formulation has been employed into MATLAB to extract natural frequency of the FGM plate with the desired accuracy. The natural frequencies calculated by DSM are compared with those available in literature.

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