Fixed point problems of nonexpansive mappings for nononvex set in Hilbert spaces

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Abstract. In this paper, we introduce a new concept of W-nonexpansive mappings and obtain fixed point theorems for nonexpansive mappings for non-convex set. Our results resolve fixed pointed problem that nonexpansive mappings be not on closed convex set, and it extends fixed point theorems for nonexpansive mappings.

Keywords: W-nonexpansive mapping, Hilbert space, fixed point theorem, non-convex set.

1. Introduction and preliminaries

Fixed point theory is widely applied in engineering. Browder (1965) [1], Kirk (1965) [2] obtained fixed points theorem for nonexpansive mapping. Non-expansion fixed point theory has made great progress, large number of results are obtained by authors (e.g. See [3-11]). let's come up with some definitions.

Definition 1.1 Let X be a nonempty set, the function $W: X \times X \to [0, \infty)$ is called triangular if for all x, $y \in X$, if $W(x,y) \geq 1$, $W(y,z) \geq 1$ or $W(y,x) \geq 1$, $W(y,z) \geq 1$, then $W(x,z) \geq 1$.

Definition 1.2 Let (X, d) be a metric space and $T: X \to X$ be a given mapping, if there exists a function $W: X \times X \to [0, \infty)$ such that $W(x,y)d(Tx,Ty) \leq d(x,y)$, $\forall x, y \in X$, then we say that T is a W-nonexpansive mapping.

Clearly, any nonexpansive mapping is a W-nonexpansive mapping with $W(x,y) = 1$ for all $x, y \in X$.

Definition 1.3 Let $T: X \to X$ be a mapping and $W: X \times X \to [0, \infty)$ be a function. We say that T is a W-admissible if $W(x,y) \geq 1 \Rightarrow W(Tx,Ty) \geq 1$, $\forall x, y \in X$.

Definition 1.4 [4] Let H be a Hilbert space, $T: H \to H$ is called demicompact if whenever $\{x_n\} \subset H$ is bounded and $\{Tx_n - Tx_n\}$ strongly convergent, then there exists a subsequence $\{x_{nk}\}$ of $\{x_n\}$ which is strongly convergent.

Next our main results are presented.

2. Main results

Theorem 2.1 Let E be a bounded closed convex subset of a Hilbert space H, $W: E \times E \to [0, \infty)$ is triangular function, $T: E \to E$ is a W-nonexpansive mapping and it is W-admissible. If the following conditions are satisfied:

(w1) there exists $x_0 \in E$ such that $W(x_0, Tx_0) \geq 1$;

(w2) there exists a sequence $\{s_j\} \subseteq [0,1)$ with $lim_{n\to\infty} s_j = 1$ such that for all $x, y \in E$, if $W(x,y) \geq 1$, then $W(x, (1-s_j)x + s_j y) \geq 1$, $\forall s_j \in \{s_j\}$;

(w3) if $\{x_n\} \subseteq E$ is satisfied $W(x_0, x_n) \geq 1$, moreover $x_n \to x^*$ or $x_n \to x^* \in E$, then $W(x_n, x*) \geq 1$.

Then T has a fixed point.

Proof. Let $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$. Take $x_{n+1j} = (1-s_j)x_0 + s_j Tx_{nj}$ for all j, $n \in N$, there $x_0 = x_{0j}$. Now we fix j, for each $j \in N$, from (w2), we may obtain $W(x_{0j}, x_{1j}) \geq 1$.

Also, for T is W-admissible, then $W(Tx_{0j}, Tx_{1j}) \geq 1$ is obtained. According to W is a triangular function and (w1), then $W(x_{0j}, Tx_{1j}) \geq 1$.

Once again use (w2), then $W(x_{0j}, x_{2j}) \geq 1$ is also obtained. Continuously, we easily obtain:

$$W(x_{0j}, x_{nj}) \geq 1, \quad \forall n \in N. \tag{1}$$

Based on that W is triangular, we may get:

$$W(x_{nj}, x_{mj}) \geq 1, \ \ \forall n, m \in N, n < m. \tag{2}$$

So from Eq. (2) and for T is W-nonexpansive, we have:

$$\begin{aligned} \|x_{nj} - x_{mj}\| &= s_j \|Tx_{n-1j} - Tx_{m-1j}\| \leq s_j W(x_{n-1j}, x_{m-1j}) \|Tx_{n-1j} - Tx_{m-1j}\| \\ &\leq s_j \|x_{n-1j} - x_{m-1j}\| \ldots \leq s_j^n \|x_{0j} - x_{m-nj}\|. \end{aligned} \tag{3}$$

Let $n \to \infty$, for E is bounded we may get $\|x_{nj} - x_{mj}\| \to 0$, hence $\{x_{nj}\}$ is Cauchy sequence, it means there exists $x_j^* \in E$ such that $\{x_{nj}\}$ convergent to x_j^*, that is:

$$\lim_{n\to\infty} \|x_{nj} - x_j^*\| = 0. \tag{4}$$

Also, from Eq. (1) and (w3), we have:

$$W(x_{nj}, x_j^*) \geq 1. \tag{5}$$

Once again by Eq. (1), for W is triangular, so we have:

$$W(x_{0j}, x_j^*) \geq 1. \tag{6}$$

Since E is bounded, closed and convex in Hilbert H, then it is weakly compact. Hence there exists a $x^* \in E$ such that:

$$x_j^* \to x^*, \quad (j \to \infty). \tag{7}$$

From Eqs. (6, 7), applying (w3) we have:

$$W(x_j^*, x^*) \geq 1. \tag{8}$$

Next, we show that $x_j^* = (1 - s_j)x_{0j} + s_j T x_j^*$.

Indeed, according to $x_{nj} = (1 - s_j)x_0 + s_j T x_{n-1j}$, T is W-nonexpiansive and Eq. (5), we have:

$$\begin{aligned} \left\|x_j^* - \left((1 - s_j)x_{0j} + s_j T x_j^*\right)\right\| &= \left\|x_j^* - x_{nj} + x_{nj} - \left((1 - s_j)x_{0j} + s_j T x_j^*\right)\right\| \\ &= \|x_{nj} - x_j^*\| + s_j \|Tx_{n-1j} - Tx_j^*\| \leq \|x_{nj} - x_j^*\| + s_j W(x_{nj}, x_j^*) \|Tx_{n-1j} - Tx_j^*\| \\ &\leq \|x_{nj} - x_j^*\| + s_j \|x_{n-1j} - x_j^*\|. \end{aligned} \tag{9}$$

Let $n \to \infty$ in Eq. (9), utilize Eq. (4) we obtain $\left\|x_j^* - \left[(1 - s_j)x_{0j} + s_j T x_j^*\right]\right\| \to 0$, it implies that $x_j^* = (1 - s_j)x_{0j} + s_j T x_j^*$.

Finally, we show that x^* is a fixed point of T. If y is any arbitrary point in H, we have:

$$\|x_j^* - y\|^2 = \|(x_j^* - x^*) + (x^* - y)\|^2 = \|x_j^* - x^*\|^2 + \|x^* - y\|^2 + 2\langle x_j^* - x^*, x^* - y\rangle. \tag{10}$$

Since $x_j^* \to x^*$, then $2\langle x_j^* - x^*, x^* - y\rangle \to 0, (j \to \infty)$.

So, based on the above inequality and Eq. (10), we get:

$$\lim_{j\to\infty}\left(\left\|x_j^* - y\right\|^2 - \left\|x_j^* - x^*\right\|^2\right) = \|x^* - y\|^2. \tag{11}$$

Setting $y = Tx^*$ in Eq. (11), we have:

$$\lim_{j\to\infty}\left(\left\|x_j^* - Tx^*\right\|^2 - \left\|x_j^* - x^*\right\|^2\right) = \|x^* - Tx^*\|^2. \tag{12}$$

Moreover, since $x_j^* = (1 - s_j)x_{0j} + s_j Tx_j^*$, then:

$$\left\|Tx_j^* - x_j^*\right\| = \left\|Tx_j^* - (1 - s_j)x_{0j} - s_j Tx_j^*\right\| = (1 - s_j)\left\|Tx_j^* - x_{0j}\right\|. \tag{13}$$

So, in Eq. (13) as $j \to \infty$, for $\lim_{j\to\infty} s_j = 1$ we have:

$$\left\|Tx_j^* - x_j^*\right\| \to 0. \tag{14}$$

On the other hand, from Eq. (8) and since T is W-nonexpansive mapping, we have:

$$\left\|Tx_j^* - Tx^*\right\| \le W(x_j^*, x^*)\left\|Tx_j^* - Tx^*\right\| \le \left\|x_j^* - x^*\right\|.$$

Thus:

$$\left\|x_j^* - Tx^*\right\| \le \left\|x_j^* - Tx_j^*\right\| + \left\|Tx_j^* - Tx^*\right\| \le \left\|x_j^* - Tx_j^*\right\| + \left\|x_j^* - x^*\right\|, \tag{15}$$

in turn:

$$\left\|x_j^* - Tx^*\right\| - \left\|x_j^* - x^*\right\| \le \left\|x_j^* - Tx_j^*\right\|. \tag{16}$$

Hence by Eq. (14), we have:

$$\lim_{j\to\infty}\left(\left\|x_j^* - Tx^*\right\| - \left\|x_j^* - x^*\right\|\right) \le \lim_{j\to\infty}\left\|x_j^* - Tx_j^*\right\| = 0. \tag{17}$$

And, due to E is bounded, we have also:

$$\begin{aligned}&\lim_{j\to\infty}\left(\left\|x_j^* - Tx_*\right\|^2 - \left\|x_j^* - x^*\right\|^2\right)\\ &\quad = \lim_{j\to\infty}\left(\left\|x_j^* - Tx_*\right\| - \left\|x_j^* - x^*\right\|\right)\left(\left\|x_j^* - Tx_*\right\| + \left\|x_j^* - x^*\right\|\right) \le 0.\end{aligned} \tag{18}$$

So, by Eq. (12), we get $\|x^* - Tx_*\|^2 = 0$, that is, x^* is fixed point of T.

Now, we provide a method for computation of that fixed point x^*.

Theorem 2.2 Suppose all conditions of the Theorem 2.1 are satisfied. Then the Krasnoselskij iteration $\{x_n\}_0^\infty$ given by:

$$x_{n+1} = (1 - s)x_n + sTx_n, \quad s \in \{s_j\}_{j\in N}, \quad n = 0,1,2, ..., \tag{19}$$

converges to a fixed point of T.

Proof. Take the same $x_0 \in E$ as Theorem 2.1, and such that $W(x_0, Tx_0) \ge 1$. From (w2) we get:

$$W(x_0,(1-s)x_0+sTx_0)=W(x_0,x_1)\geq 1. \tag{20}$$

For W is triangular, so:

$$W(Tx_0,x_1)\geq 1. \tag{21}$$

Since T is a W-admissible, from Eq. (20) we have:

$$W(Tx_0,Tx_1)\geq 1. \tag{22}$$

Once again for W is triangular, by Eqs. (21) and (22) we have $W(x_1,Tx_1)\geq 1$.
Also, from (w2) we have $W(x_1,(1-s)x_1+sTx_1)=W(x_1,x_2)\geq 1$.
Continuously, we can obtain:

$$W(x_n,x_m)\geq 1,\ \ \forall n,m\in N,\quad n<m. \tag{23}$$

Hence:

$$W(x_0,x_n)\geq 1. \tag{24}$$

Also form Theorem 2.1, we know that x_* is fixed point of T, and Based on all conditions of Theorem 2.1 are satisfied in Theorem 2.2, similarly we have:

$$W\left(x_0,x_j^*\right)\geq 1, \tag{25}$$
$$W\left(x_j^*,x^*\right)\geq 1. \tag{26}$$

From Eqs. (25) and (26), for W is triangular, then:

$$W(x_0,x^*)\geq 1. \tag{27}$$

Also, by Eqs. (24) and (27), use W is triangular, we get:

$$W(x_n,x^*)\geq 1. \tag{28}$$

Based on Eq. (28), since T is W-nonexpansive mapping, then we have:

$$\|Tx_n-Tx^*\|\leq W(x_n,x^*)\|Tx_n-Tx^*\|\leq\|x_n-x^*\|. \tag{29}$$

So:

$$\begin{aligned}\|x_{n+1}-x^*\|&=\|(1-s)(x_n-x^*)+s(Tx_n-Tx^*)\|\\&\leq(1-s)\|(x_n-x^*)\|+s\|Tx_n-Tx^*\|\\&\leq(1-s)\|(x_n-x^*)\|+s\|x_n-x^*\|=\|(x_n-x^*)\|.\end{aligned} \tag{30}$$

Continuously, we have $\|x_{n+1}-x^*\|\leq\|x_0-x^*\|$, which implies that $\{\|x_{n+1}-x^*\|\}$ is monotone decrease bounded sequence. So $\lim_{n\to\infty}\|x_{n+1}-x^*\|$ exists.
Next, we prove that $\|x_n-Tx_n\|\to 0$:

$$\begin{aligned}\|x_{n+1}-x^*\|^2&=\|(1-s)(x_n-x^*)-s(Tx_n-Tx^*)\|^2\\&=(1-s)^2\|x_n-x^*\|^2+s^2\|Tx_n-Tx^*\|^2+2(1-s)s\langle x_n-x^*,Tx_n-Tx^*\rangle\\&\leq(1-s)^2\|x_n-x^*\|^2+s^2\|x_n-x^*\|^2+2(1-s)s\langle x_n-x^*,Tx_n-Tx^*\rangle\\&=((1-s)^2+s^2)\|x_n-x^*\|^2+2(1-s)s\langle x_n-x^*,Tx_n-Tx^*\rangle.\end{aligned} \tag{31}$$

Also, on the other hand for any constant λ:

$$\begin{aligned}\lambda^2\|x_n - Tx_n\|^2 &= \|(x_n - x^*) - (Tx_n - Tx^*)\|^2 \\ &= \lambda^2\|x_n - x^*\|^2 + \lambda^2\|Tx_n - Tx^*\|^2 - 2\lambda^2\langle x_n - x^*, Tx_n - Tx^*\rangle \\ &\leq \lambda^2\|x_n - x^*\|^2 + \lambda^2\|x_n - x^*\|^2 - 2\lambda^2\langle x_n - x^*, Tx_n - Tx^*\rangle \\ &= 2\lambda^2\|x_n - x^*\|^2 - 2\lambda^2\langle x_n - x^*, Tx_n - Tx^*\rangle \end{aligned} \tag{32}$$

Adding Eq. (31) to Eq. (32) and let $\lambda^2 \leq (1-s)s$, we may obtain:

$$\begin{aligned}\|x_{n+1} - x^*\|^2 + \lambda^2\|x_n - Tx_n\|^2 &\leq ((1-s)^2 + s^2 + 2\lambda^2)\|x_n - x^*\|^2 \\ &+(2(1-s)s - 2\lambda^2)\langle x_n - x^*, Tx_n - Tx^*\rangle \\ &\leq ((1-s)^2 + s^2 + 2\lambda^2 + 2(1-s)s - 2\lambda^2)\|x_n - x^*\|^2 = \|x_n - x^*\|^2. \end{aligned} \tag{33}$$

It implies $\lambda^2\|x_n - Tx_n\|^2 \leq \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2$.

Since $\lim_{n\to\infty}\|x_{n+1} - x^*\|$ exists, in the above inequality let $n \to \infty$, it results $\lambda^2\|x_n - Tx_n\|^2 \to 0$.

It means $\|x_n - Tx_n\| \to 0$.

For T is demicompact, it results that there exists a strongly convergent subsequence $\{x_{n_i}\} \subseteq \{x_n\}$ such that $x_{n_i} \to x^* \in F(T)$, that is, $\|x_{n_i} \to x^*\| \to 0$. Also $\{\|x_n \to x^*\|\}$ is convergent, it implies that $\|x_n \to x^*\| \to 0$. Hence that $\{x_n\}$ is convergent to $x^* \in F(T)$.

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