49. Mathematical solutions for the flexural analysis of Mindlin's first order shear deformable circular plates

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Abstract. In this work, the problem of first order shear deformable solid circular plate under transverse load was solved mathematically. The problem considered was assumed axisymmetric. The plate and loading were considered axisymmetric. The problem was defined as a boundary value problem of a system of differential equations of equilibrium in terms of the stress resultants and the stress – resultants – displacement relations. The set of equations were considered simultaneously to express them in variable separable form. The mathematical technique of separation of variables was then used to obtain solutions for the unknown generalised displacements. Specific problems of clamped edge plates and simply supported edge plates under uniformly distributed load and point load at the centre were considered and solved using the same technique of separation of variables. The mathematical expressions obtained showed that in all cases, the deflection was expressible in terms of flexural and shear components. The maximum deflection was found to occur at the plate centre as is expected from the symmetrical nature of the problem. The shear component of the transverse deflection was found to significantly increase with significant increase in the ratio of the plate thickness to the radius (h/r_0).

1. Introduction

Circular plate structures have extensive applications in aerospace, civil, structural, geotechnical, naval, and mechanical engineering [1]. They can be subjected to distributed transverse loads, in-plane loads or dynamic loads. They are classified according to the ratio of their thickness, h to the diameter, D, [2, 3] as: thin plates ($D/h > 100$), moderately thick plates ($20 < D/h < 100$), and thick plates ($D/h < 3$). Circular plates are also classified according to their material properties as: laminated plates, anisotropic plates, orthotropic plates, homogeneous plates, heterogeneous plates and isotropic plates [4-8].

The determination of displacement and deflection functions in elastic circular plates subjected to symmetrical distribution of transverse loading is a solid mechanics/theory of elasticity problem frequently encountered in the analysis and design of structural elements and systems [9]. Plate problems are generally three dimensional (3D) problems of the mathematical theory of elasticity, and the three spatial coordinates are used in their formulation. Three-dimensional plate theories have their mathematical and analytical foundations in the mathematical theory of elasticity; and the solution for 3D problems are mathematical rigorous and demanding. Only few 3D problems have so far been solved [10-14].

Theories that have been used to describe the flexural problem of circular plates include: Kirchhoff-Love plate theory also called the Classical small deformation thin plate theory (CPT) [15-18], Mindlin plate theory [19], Reissner plate theory [20-22], Von Karman plate theory [23], Shimpi's plate theory [24, 25], Levinson's plate theory [26], Reddy's plate theory [27, 28] and Modified/refined plate theories [29].

The Kirchhoff-Love plate theory is based on the Kirchhoff-Love's hypotheses which are assumptions similar to the Bernoulli-Navier's hypotheses used in the classical theory of thin (or

slender) beams. The fundamental assumptions of the classical small deflection thin plate theory for homogeneous, isotropic elastic plates are [18]:

(i) Straight line initially normal to the middle surface to the plate remains straight and normal to the deformed middle surface of the plate and unchanged in length.

(ii) The transverse displacement w is assumed to be very small. This implies that the slope of the deflection surface is small and hence the square of the slope would be considered negligible in comparison with unity.

(iii) The normal stresses in the radial and tangential (circumferential) directions σ_{rr} and $\sigma_{\theta\theta}$ and the in-plane shear stress $\tau_{r\theta}$ are assumed to be zero at the middle surface for small deflection cases (i.e. $w \ll h$).

(iv) The transverse normal stress σ_{zz} is very small as compared to the other stress components and can be disregarded in the Hooke's generalised stress-strain relations without introducing significant errors i.e. $\sigma_{zz} \ll (\sigma_{rr}, \sigma_{\theta\theta}, \tau_{r\theta})$.

(v) The middle plane remains unstrained after bending deformation and is a neutral surface (plane).

The above assumptions reduce the plate problem from a 3D mathematical theory of elasticity problem to a two dimensional (2D) approximate theory. Hence only two normal stresses σ_{rr}, $\sigma_{\theta\theta}$ and one in-plane transverse shear stress $\tau_{r\theta}$ are the stresses used to define the plate problem. These stresses are functions of the two coordinate variables r, and θ.

The differential equations of equilibrium of circular plates are [6]:

$$\frac{\partial M_{rr}}{\partial r} + \frac{1}{r}\frac{\partial M_{r\theta}}{\partial \theta} + \frac{M_{rr} - M_{\theta\theta}}{r} - Q_r = 0, \tag{1}$$

$$\frac{\partial M_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial M_{\theta\theta}}{\partial \theta} + \frac{2M_{r\theta}}{r} - Q_\theta = 0, \tag{2}$$

$$\frac{\partial Q_r}{\partial r} + \frac{1}{r}\frac{\partial Q_\theta}{\partial \theta} + \frac{Q_r}{r} + q = 0, \tag{3}$$

where M_{rr} are radial moments, $M_{\theta\theta}$ are circumferential moments, $M_{r\theta}$ is the twisting moment, Q_r, Q_θ are shear force in the r and θ directions, Q_r is the radial shear force, Q_θ is the circumferential shear force and q is the transverse load distribution.

The differential equation of equilibrium is expressed in terms of the deflection of the middle surface by the fourth order biharmonic equation:

$$\nabla^4 w = \nabla^2 \nabla^2 w = \frac{q}{D}, \tag{4}$$

where:

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial \theta^2}, \tag{5}$$

and ∇^2 is the Laplacian operator in the polar coordinate system, ∇^4 is the biharmonic operator. D is the modulus of flexural rigidity of the plate.

For cases of axisymmetric or rotationally symmetric loading, the load, internal stress resultants (bending moment and shear force distributions) and the displacements become functions of one coordinate variable only, namely, r. In such cases, each vertical plane (rz) of the plate will be a principal plane, and hence the twisting moment $M_{r\theta}$ will vanish [6]. Also, the shear force Q_θ and the tangential displacement v will be zero. The differential equation of equilibrium of axisymmetric circular Kirchhoff-Love plate will simplify to [30, 31]:

$$\frac{dM_{rr}}{dr} + \frac{M_{rr} - M_{\theta\theta}}{r} - Q_r = 0, \tag{6}$$

$$\frac{dQ_r}{dr} + \frac{Q_r}{r} + q = 0. \tag{7}$$

The moment-curvature (moment-deflection) equations become:

$$M_{rr} = -D\left(\frac{d^2w}{dr^2} + \frac{\mu}{r}\frac{dw}{dr}\right), \tag{8}$$

$$M_{\theta\theta} = -D\left(\mu\frac{d^2w}{dr^2} + \frac{1}{r}\frac{dw}{dr}\right). \tag{9}$$

The radial shear force Q_r is:

$$Q_r = -D\frac{d}{dr}\nabla^2 w. \tag{10}$$

D is related to the Young's modulus, E and the Poisson's ratio, μ by:

$$D = \frac{Eh^3}{12(1 - \mu^2)}, \tag{11}$$

where h is the plate thickness.

∇^2 is the Laplacian in axisymmetric polar coordinates:

$$\nabla^2 = \frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr}. \tag{12}$$

Kirchhoff's theory, through satisfactory for thin plates, has been found to be unsatisfactory as the plate thickness increases. Its major limitation is the disregard for shear deformation which renders it incapable of accurately describing the behaviour of moderately thick plates; where shear deformation will be significant. Von Karman's theory for the large deflection, axisymmetric bending of a circular plate which is either simply supported or clamped at the boundary can be represented by the system of coupled non-linear partial differential equations [23] expressed in terms of two unknowns $w(r)$ and $\phi(r)$:

$$\frac{Eh^3}{12(1 - \mu^2)}\nabla^2 w - \frac{1}{r}\frac{d}{dr}\left(\frac{d\phi}{dr}\frac{dw}{dr}\right) = \frac{p(r)}{h}, \tag{13}$$

$$\frac{1}{E}\nabla^2\phi + \frac{1}{r}\frac{dw}{dr}\frac{d^2w}{dr^2} = 0, \tag{14}$$

where:

$$\nabla^2 = \frac{1}{r}\frac{d}{dr}r\frac{d}{dr}, \tag{15}$$

and E and μ are the Young's modulus and Poisson's ratio, respectively. h is the plate thickness, r is the radial coordinate variable. ϕ is the Airy stress function, $p(r)$ is the applied radially symmetric load distribution.

The study is focused on first order shear deformable circular plate theory of Mindlin. The theory is capable of describing the behaviours of moderately thick plates where shear deformations will contribute to the flexural behaviour.

1.1. Review of solution methods for the boundary value problem of circular plates

The boundary value problem (BVP) of circular plates is represented generally by differential equations and boundary conditions. They are solved using techniques for solving ordinary and partial differential equations. The problems can also be formulated in weighted residual form or using variational calculus. Thus, the methods used in the literature for the solution of the plate problem include: Separation of variables, Integral transforms, Eigen function expansions, Fourier series, etc. The numerical methods used in the literature include: Galerkin method, Kantorovich method, Ritz method, Finite Element method and boundary element methods [32-35].

1.2. Justification for the Mindlin First order shear deformable theory

The classical two dimensional (2D) Kirchhoff – theory of flexure of elastic circular (thin) plates disregards the axial shear strains and normal stress. Hence the Kirchhoff circular plate theory cannot be expected to give good estimates of the deflection and internal forces in circular plates with a thickness to radius ratio exceeding 0.05. Mindlin [19] considered the effect of rotatory inertia and shear deformation on the flexural behaviour of isotropic circular plates and derived from first principles the governing equations for a more comprehensive 2D theory called the Mindlin first order shear deformation plate theory FSDT for circular plates.

Mindlin's theory is analogous to the Timoshenko's theory for beams, which takes into consideration shear deformation in the flexural analysis of beams. Mindlin's FSDT for circular plates satisfies the stress-strain equations for transverse shear stresses and shear strains by the use of shear modification (correction) factors k_s. The shear modification factors depend upon the plate material, geometry, cross-sectional shape, loading, and support/boundary conditions.

However, a fundamental inadequacy of the Mindlin's FSDT is that all the equations of three dimensional theory of elasticity are not satisfied by the theory, and only deflections of the middle surfaces (planes)of the plates are considered [36]. Despite the obvious limitation, the Mindlin 2D plate theory plays a significant role in the validation of results for deflections and internal stress resultants of thick plate theories [36].

Reddy's higher order shear deformation plate theory (HSDPT) assumes a quadratic (parabolic) variation of transverse shear stress through the plate thickness such that shear stress free boundary conditions required for the top and bottom surfaces of the plate are automatically satisfied. Hence the Reddy theory does not require shear correction factors.

1.3. Advantages of mathematical solutions over numerical solutions in plate bending analysis

The focus of the present study is to derive mathematical solutions, also called closed form or analytical solutions to the flexural problem of Mindlin circular plates; due to the obvious merits of the mathematical solutions over the numerical or approximate solutions. The advantages of the mathematical solutions are:

(i) The mathematical solutions aim to derive using rigorous mathematical techniques, the "exact" solutions to the governing partial differential equations of the Mindlin circular plate theory, which are valid at all points on the circular plate domain as well as on the boundaries; for given loading and restraint conditions. The solutions derived in a mathematical approach are called 'exact' within the limitations of the foundational theory used in formulating the plate problem, in this case, the Mindlin first order shear deformation theory. However, the numerical solutions aim to obtain approximate solutions to the plate flexure problem which may not be valid solutions at all points on the solution domain; nor at the boundaries.

(ii) The derivation of analytical solutions is important since they serve as benchmark solutions for comparing the accuracy of numerical solutions.

(iii) The advantages of mathematical solutions include its better accuracy as compared with

numerical solutions and broad generality in solving plate flexure problems with given loads and restraint conditions.

(iv) The mathematical derivation of the solutions provide a systematic framework for solving the flexural problems of Mindlin circular plates with different edge restraint conditions.

Despite the advantages of mathematical solutions, exact analytical solutions for plate bending problems are very limited, even for the classical Kirchhoff-thin plate theory. They are usually mathematically rigorous, and mathematical solutions do not exist for many plate bending problems involving complicated restraint conditions and complex load distributions. For plates of arbitrary shape, the solution of the governing plate equations, even in the classical Kirchhoff thin plate theory is a non-trivial matter and in general, an exact analytical/mathematical solution does not exist [37].

Even for the classical Kirchhoff plate theory, the mathematical rigours of obtaining closed form solutions to the BVP are so complex that mathematical solutions leading to formula for stresses or deflections are only possible/available for a few simple cases of geometry, boundary conditions and loading [38].

1.4. Research aim and objectives

The general aim of this paper is to solve and obtain mathematical solutions for the boundary value problem of first order shear deformable circular plates. The general objectives are:

(i) to solve the governing boundary value problem of flexure of first order shear deformable circular plates under arbitrary distributed load $p(r)$ and obtain general solutions for the generalised displacements in terms of constants of integration.

(ii) to find mathematical solutions to the flexural problem of first order shear deformable circular plates with clamped edge ($r = r_0$) for the case of (a) uniformly distributed transverse load over the entire plate domain, and (b) point load P_0 applied at the centre ($r = 0$) of the plate.

(iii) to find analytical solutions for the flexural problem of first order shear deformable circular plates with simply supported edges for the case of uniformly distributed transverse load over the entire plate domain.

2. Theoretical framework

Axisymmetric problems of the theory of elasticity are governed by the requirements of the strain displacement relations, the differential equations of equilibrium and the stress-strain law. The differential equations of equilibrium are given by:

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r}\frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{1}{r}(\sigma_{rr} - \sigma_{\theta\theta}) = 0, \tag{16}$$

$$\frac{\partial \tau_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \tau_{\theta\theta}}{\partial \theta} + \frac{2\tau_{r\theta}}{r} = 0, \tag{17}$$

where σ_{rr} and $\sigma_{\theta\theta}$ are the radial and circumferential stresses and $\tau_{r\theta}$ is the shear stress.

The strain-displacement relations for axisymmetric small displacement problems are given by the three equations:

$$\varepsilon_{rr} = \frac{\partial u_r}{\partial r}, \tag{18}$$

$$\varepsilon_{\theta\theta} = \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r}, \tag{19}$$

$$2\varepsilon_{r\theta} = \gamma_{r\theta} = \frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}, \tag{20}$$

where ε_{rr} is the radial strain (normal strain in the radial direction), $\varepsilon_{\theta\theta}$ is the circumferential

normal strain, $\varepsilon_{r\theta}$ is the shear strain in the $r\theta$ direction, u_r is the radial component of displacement, u_θ is the circumferential component (tangential component) of the displacement.

The strain-stress relations in the axisymmetric (polar) coordinates are for plane stress conditions:

$$\varepsilon_{rr} = \frac{1}{E}(\sigma_{rr} - \mu\sigma_{\theta\theta}), \tag{21}$$

$$\varepsilon_{\theta\theta} = \frac{1}{E}(\sigma_{\theta\theta} - \mu\sigma_{rr}), \tag{22}$$

$$\varepsilon_{zz} = -\frac{\mu}{E}(\sigma_{rr} + \sigma_{\theta\theta}), \tag{23}$$

$$2\varepsilon_{r\theta} = \frac{2(1+\mu)}{E}\tau_{r\theta} = \frac{\tau_{r\theta}}{G} = \gamma_{r\theta}, \tag{24}$$

and for plane strain conditions:

$$\varepsilon_{rr} = \frac{1+\mu}{E}((1-\mu)\sigma_{rr} - \mu\sigma_{\theta\theta}), \tag{25}$$

$$\varepsilon_{\theta\theta} = \frac{1+\mu}{E}(\mu\sigma_{rr} + (1-\mu)\sigma_{\theta\theta}), \tag{26}$$

$$\gamma_{r\theta} = \frac{\tau_{r\theta}}{G} = \frac{2(1+\mu)\tau_{r\theta}}{E}, \tag{27}$$

where E is the Young's modulus of elasticity, μ is the Poisson's ratio, and G is the shear modulus. G is related to the Young's modulus E by:

$$G = \frac{E}{2(1+\mu)}. \tag{28}$$

3. Governing equations

The differential equations of equilibrium of first order shear deformable circular plates are given by the system of three partial differential equations in terms of the stress resultants, M_{rr}, $M_{\theta\theta}$, $M_{r\theta}$, Q_r and Q_θ as follows:

$$-\frac{1}{r}\left(\frac{\partial}{\partial r}(rM_{rr}) + \frac{\partial M_{r\theta}}{\partial\theta} - M_{\theta\theta}\right) + Q_r = 0, \tag{29}$$

$$-\frac{1}{r}\left(\frac{\partial}{\partial r}(rM_{r\theta}) + \frac{\partial M_{\theta\theta}}{\partial\theta} + M_{r\theta}\right) + Q_\theta = 0, \tag{30}$$

$$-\frac{1}{r}\left(\frac{\partial}{\partial r}(rQ_r) + \frac{\partial\theta_\theta}{\partial\theta}\right) - p = 0. \tag{31}$$

The stress resultants for first order shear deformable circular plates are expressed in terms of the displacements w_0, φ_r and φ_θ as:

$$M_{rr} = D\left(\frac{\partial\varphi_r}{\partial r} + \frac{\mu}{r}\left(\varphi_r + \frac{1}{r}\frac{\partial\varphi_\theta}{\partial\theta}\right)\right), \tag{32}$$

$$M_{\theta\theta} = D\left(\mu\frac{\partial\varphi_r}{\partial r} + \frac{1}{r}\left(\varphi_r + \frac{1}{r}\frac{\partial\varphi_\theta}{\partial\theta}\right)\right), \tag{33}$$

$$M_{r\theta} = \frac{D(1-\mu)}{r}\left(\frac{\partial\varphi_r}{\partial\theta} + r\frac{\partial\varphi_\theta}{\partial r} - \varphi_\theta\right), \tag{34}$$

$$Q_r = k_s \int_{-h/2}^{h/2} \tau_{rz} \, dz = k_s Gh \left(\varphi_r + \frac{\partial w_0}{\partial r} \right), \tag{35}$$

$$Q_\theta = k_s \int_{-h/2}^{h/2} \tau_{\theta z} \, dz = k_s Gh \left(\varphi_\theta + \frac{1}{r} \frac{\partial w_0}{\partial \theta} \right), \tag{36}$$

where k_s is the shear correction factor.

For axisymmetric problems:

$$\varphi_\theta = 0, \tag{37}$$
$$M_{r\theta} = 0, \tag{38}$$
$$Q_\theta = 0. \tag{39}$$

Consequently, the differential equations of equilibrium of axially symmetric problems simplify to the system of two partial differential equations:

$$-\frac{1}{r} \left(\frac{\partial}{\partial r} (r M_{rr}) - M_{\theta\theta} \right) + Q_r = 0, \tag{40}$$

$$-\frac{1}{r} \frac{d}{dr} (r Q_r) - p(r) = 0. \tag{41}$$

For axially symmetric problems of first order shear deformable circular plates, the internal stress resultants (bending moments M_{rr}, $M_{\theta\theta}$ and shear force distributions Q_r) are related to the generalized displacement functions φ_r, w_0 as follows:

$$M_{rr} = D \left(\frac{d\varphi_r}{dr} + \mu \frac{\varphi_r}{r} \right), \tag{42}$$

$$M_{\theta\theta} = D \left(\mu \frac{d\varphi_r}{dr} + \frac{\varphi_r}{r} \right), \tag{43}$$

$$Q_r = k_s Gh \left(\varphi_r + \frac{dw_0}{dr} \right). \tag{44}$$

4. General solution of the boundary value problem of first order shear deformable circular plates under transverse flexure

The differential equation of equilibrium Eq. (41) is observed to be variable separable. We thus seek a solution using the method of separation of variables. Eq. (41) is rewritten as:

$$-\frac{1}{r} \frac{d}{dr} (r \, Q_r) = p(r). \tag{45}$$

Separation of variables, and integration yields:

$$\int d(r Q_r) = r Q_r = -\int r \, p(r) dr + c_0, \tag{46}$$

where c_0 is a constant of integration.

From Eq. (40), we have:

$$\frac{\partial}{\partial r} (r \, M_{rr}) - M_{\theta\theta} = r \frac{dM_{rr}}{dr} + M_{rr} - M_{\theta\theta} = r Q_r. \tag{47}$$

Substitution of Eqs. (42) and (43) into Eq. (47) yields after simplification:

$$D\left(r\frac{d^2\varphi_r}{dr^2} + \frac{d\varphi_r}{dr} - \frac{\varphi_r}{r}\right) = rQ_r = D\left(r\frac{d}{dr}\left(\frac{d\varphi_r}{dr} + \frac{\varphi_r}{r}\right)\right) = rQ_r = Dr\left(\frac{1}{r}\frac{d}{dr}(r\varphi_r)\right). \tag{48}$$

Using Eq. (46) we have:

$$Dr\frac{d}{dr}\left(\frac{d\varphi_r}{dr} + \frac{\varphi_r}{r}\right) = -\int rp(r)dr + c_0. \tag{49}$$

Solution by successive integration yields:

$$D\varphi_r = -\frac{1}{r}\int r\int\frac{1}{r}\int r\,p(r)drdrdr + \frac{c_0 r}{4}(2\log r - 1) + \frac{rc_1}{2} + \frac{c_2}{r}, \tag{50}$$

where c_1 and c_2 are constants of integration.

Thus, we obtain:

$$D\varphi_r = -I'(r) + \frac{r}{4}(2\log r - 1)c_0 + \frac{r}{2}c_1 + \frac{c_2}{r}, \tag{51}$$

where:

$$I'(r) = \frac{1}{r}\int r\int\frac{1}{r}\int r\,p(r)drdrdr, \tag{52}$$

$$I_r(r) = \int\frac{1}{r}\int r\int\frac{1}{r}\int r\,p(r)drdrdr\,dr. \tag{53}$$

The solution for w_0 is obtained using Eqs. (44) and (46).

Thus:

$$r\,Q_r = rk_sGh\left(\varphi_r + \frac{dw_0}{dr}\right) = -\int rp(r)dr + c_0, \tag{54}$$

$$k_sGh\frac{dw_0}{dr} = -\frac{1}{r}\int rp(r)dr + \frac{c_0}{r} - k_sGh\varphi_r, \tag{55}$$

$$k_sGh\frac{dw_0}{dr} = -\frac{1}{r}\int rp(r)dr + \frac{c_0}{r} - \frac{k_sGh}{D}\left(-I'(r) + \frac{c_0 r}{4}(2\log r - 1) + \frac{rc_1}{2} + \frac{c_2}{r}\right). \tag{56}$$

Integrating both sides of Eq. (56) with respect to r, we obtain:

$$k_sGhw_0(r) = -\int\frac{1}{r}\int rp(r)drdr$$
$$+ \int\frac{c_0}{r}dr - \frac{k_sGh}{D}\int\left(-I'(r) + \frac{c_0 r}{4}(2\log r - 1) + \frac{c_1 r}{2} + \frac{c_2}{r}\right)dr\,. \tag{57}$$

Thus:

$$k_sGhw_0(r) = -\int\frac{1}{r}\int r\,p(r)drdr + c_0\log r$$
$$- \frac{k_sGh}{D}\left(-I(r) + \frac{c_0 r^2}{4}(\log r - 1) + \frac{r^2 c_1}{4} + c_2\log r\right) + c_3, \tag{58}$$

where c_3 is a fourth constant of integration.

The equations obtained for $\varphi_0(r)$ and $w_0(r)$ given respectively by Eqs. (51) and (58) are the general solutions for the boundary value problem of first order shear deformable circular plates subject to the transverse distributed load $p(r)$. The general solution is obtained in terms of four unknown constants of integration c_0, c_1, c_2 and c_3. These four constants of integration can be determined from the essential and natural boundary conditions for particular problems of first order shear deformable circular plates.

This paper considers three particular problems/cases, namely: (i) solid first order shear deformable circular plate with clamped edges and under uniformly distributed transverse load (ii) solid first order shear deformable circular plate with clamped edges and under point load P_0 applied at the centre (iii) simply supported solid shear deformable circular plate under uniformly distributed load p_0.

4.1. Case 1: First order shear deformable solid circular plate with clamped edge and under uniform transverse load distribution

For bounded solutions for the rotation φ_r, φ_r is required to be non-singular at $r = 0$, hence:

$$c_2 = 0. \tag{59}$$

For solid circular plates that are not subject to point load at the centre, $r = 0$, the shear force at the centre is required to vanish. This yields from Eq. (46):

$$c_0 = 0. \tag{60}$$

For clamped solid circular first order shear deformable plates of radius r_0, the boundary conditions of the clamped edge $r = r_0$ are:

$$w_0(r = r_0) = 0, \tag{61}$$
$$\varphi_r(r = r_0) = 0. \tag{62}$$

For uniformly distributed transverse load of intensity p_0, $p(r) = p_0$:

$$I(r) = \int \frac{1}{r} \int r \int \frac{1}{r} \int r p_0 \, dr dr dr dr = p_0 \int \frac{1}{r} \int r \int \frac{1}{r} \int r \, dr dr dr dr = \frac{p_0 r^4}{64}, \tag{63}$$
$$I'(r) = \frac{p_0 r^3}{16}. \tag{64}$$

The solution for $w_0(r)$ becomes:

$$k_s G h w_0(r) = -\int \frac{1}{r} \int r \, p_0 \, dr dr - \frac{k_s G h}{D} \left(\frac{p_0 r^4}{64} + \frac{r^2 c_1}{4} \right) + c_3, \tag{65}$$
$$k_s G h w_0(r) = \frac{-p_0 r^2}{4} - \frac{k_s G h}{D} \left(\frac{p_0 r^4}{64} + \frac{r^2 c_1}{4} \right) + c_3. \tag{66}$$

Similarly, the solution for φ_r becomes:

$$D \varphi_r = \frac{-p_0 r^3}{16} + \frac{r c_1}{2}. \tag{67}$$

Using the boundary condition Eq. (62) in Eq. (67) we obtain:

$$c_1 = \frac{p_0 r_0^2}{8}. \tag{68}$$

Application of the boundary condition Eq. (61) in Eq. (66) yields:

$$c_3 = \frac{p_0 r_0^2}{4} + \frac{k_s G h}{D} \frac{p_0 r_0^4}{64}. \tag{69}$$

Substitution of c_1 and c_3 into the equation for $w_0(r)$ and φ_r give the solutions for this case as:

$$
\begin{aligned}
w_0(r) &= \frac{p_0 r_0^4}{64D}\left(1 - \left(\frac{r}{r_0}\right)^2\right)^2 + \frac{p_0 r_0^2}{4k_s G h}\left(1 - \left(\frac{r}{r_0}\right)^2\right) \\
&= \frac{p_0 r_0^4}{64D}\left(1 - 2\left(\frac{r}{r_0}\right)^2 + \left(\frac{r}{r_0}\right)^4\right) + \frac{p_0 r_0^2}{4k_s G h}\left(1 - \left(\frac{r}{r_0}\right)^2\right),
\end{aligned}
\tag{70}
$$

$$\varphi_r(r) = \frac{p_0 r_0^2 r}{16D}\left(1 - \left(\frac{r}{r_0}\right)^2\right), \tag{71}$$

$$\varphi_r(r) = \frac{p_0 r_0^3}{16D}\left(\frac{r}{r_0}\right)\left(1 - \left(\frac{r}{r_0}\right)^2\right) = \frac{p_0 r_0^3}{16D}\left(\frac{r}{r_0} - \left(\frac{r}{r_0}\right)^3\right). \tag{72}$$

The transverse deflection $w_0(r)$ can be decomposed into two components, a flexural (bending) component $w_0^f(r)$ and a shear component $w_0^s(r)$ as follows:

$$w(r) = w_0^f(r) + w_0^s(r), \tag{73}$$

where:

$$w_0^f(r) = \frac{p_0 r_0^4}{64D}\left(1 - \left(\frac{r}{r_0}\right)^2\right)^2, \tag{74}$$

$$w_0^s(r) = \frac{p_0 r_0^2}{4k_s G h}\left(1 - \left(\frac{r}{r_0}\right)^2\right). \tag{75}$$

4.1.1. Maximum deflection

The maximum deflection $w_{\max}$ is found to occur at the centre of the plate where $r = 0$, and is given by:

$$w_{\max} = \frac{p_0 r_0^4}{64D} + \frac{p_0 r_0^2}{4k_s G h}, \tag{76}$$

$$w_{\max} = \frac{p_0 r_0^4}{64D}\left(1 + \frac{8}{3k_s(1-\mu)}\left(\frac{h}{r_0}\right)^2\right). \tag{77}$$

For $k_s = 5/6$, and $\mu = 0.30$:

$$w_{\max} = \frac{p_0 r_0^4}{64D}\left(1 + \frac{8}{1.75}\left(\frac{h}{r_0}\right)^2\right). \tag{78}$$

Table 1 shows the maximum deflection of first order shear deformable solid circular plates with clamped edge for the case of uniformly distributed load computed for varying values of h/r_0. Table 2 shows the difference between $w_{\max}$ the maximum deflection obtained by the first

order shear deformable circular plate theory and the classical Kirchhoff-Love circular plate theory for various values of h/r_0 for the case of clamped edge and uniformly distributed load.

Table 1. Maximum deflection of first order shear deformable circular plates
with clamped edges and under uniformly distributed load p_0

$\dfrac{h}{r_0}$	$w_{\max}\left(\times \dfrac{p_0 r_0^4}{D}\right)$
0.005	1.562679×10^{-2}
0.01	1.563214×10^{-2}
0.05	1.580357×10^{-2}
0.10	1.633929×10^{-2}
0.2	1.848214×10^{-2}
0.3	2.220536×10^{-2}

Table 2. Difference between the maximum deflections of first order shear deformable circular plates
and the Kirchhoff-Love circular plate with clamped edges (case of uniformly distributed load)

h/r_0	$(w_{\max}^{FSDP} - w_{\max}^{CPT})$	Difference %
0.005	1.142857×10^{-4}	0.0114
0.01	4.57142×10^{-4}	0.0457
0.05	1.142857×10^{-2}	1.1429
0.1	4.57142×10^{-2}	4.5714
0.2	18.2857×10^{-2}	18.2857
0.3	41.142857×10^{-2}	41.1429

The bending moment and shear force distributions are found by substitution of Eq. (72) into Eqs. (42) and (43) as:

$$M_{rr}(r) = \frac{p_0 r_0^2}{16}\left[(1+\mu) - (3+\mu)\left(\frac{r}{r_0}\right)^2\right], \tag{79}$$

$$M_{\theta\theta}(r) = \frac{p_0 r_0^2}{16}\left[(1+\mu) - (1+3\mu)\left(\frac{r}{r_0}\right)^2\right]. \tag{80}$$

The bending stresses are found as:

$$\sigma_{rr}(r,z) = \frac{3p_0 r_0^2 z}{4h^3}\left[(1+\mu) - (3+\mu)\left(\frac{r}{r_0}\right)^2\right]. \tag{81}$$

$$\sigma_{\theta\theta}(r,z) = \frac{3p_0 r_0^2 z}{4h^3}\left[(1+\mu) - (1+3\mu)\left(\frac{r}{r_0}\right)^2\right]. \tag{82}$$

The shear stress distribution τ_{rz} is given by:

$$
\begin{aligned}
\tau_{rz}(r) &= G\gamma_{rz} = G\left(\varphi_r + \frac{dw_0(r)}{dr}\right) \\
&= G\left(\frac{p_0 r_0^4}{64D}\left(\frac{-4r}{r_0^2} + \frac{4r^3}{r_0^4}\right) + \frac{p_0 r_0^2}{4k_s Gh}\left(-\frac{2r}{r_0^2}\right) + \frac{p_0 r_0^3}{16D}\left(\frac{r}{r_0} - \left(\frac{r}{r_0}\right)^3\right)\right) \\
&= G\left(\frac{p_0 r_0^3}{16D}\left(\frac{-r}{r_0} + \left(\frac{r}{r_0}\right)^3 + \left(\frac{r}{r_0}\right) - \left(\frac{r}{r_0}\right)^3\right) - \frac{p_0 r}{2k_s Gh}\right) = -\frac{p_0 r}{2k_s h}.
\end{aligned} \tag{83}
$$

The shear force distribution $Q_r(r)$ is from Eq. (75):

$$Q_r(r) = k_s Gh \frac{dw_0^s}{dr} = k_s Gh \frac{p_0 r_0^2}{4k_s Gh}\left(\frac{-2r}{r_0^2}\right) = -\frac{p_0 r}{2} = k_s h\, \tau_{rz} = k \int_{-h/2}^{h/2} \tau_{rz}\, dz. \tag{84}$$

The deflection and rotation functions satisfy the boundary conditions at the clamped edge since upon substitution of $r = r_0$ into the equations, Eqs. (61) and (62) are satisfied. They also satisfy the conditions of symmetry since $\varphi_r(r = 0) = 0$.

4.2. Case 2: First order shear deformable solid circular plate with clamped edge and point load P_0 applied at the centre, $r = 0$

For first order shear deformable solid circular plate (FSDP) subject to a point load at the centre, enforcement of boundary conditions yield:

$$c_0 = -\frac{P_0}{2\pi}. \tag{85}$$

The requirement of boundedness of $w_0(r)$ also require that:

$$c_2 = 0. \tag{86}$$

Then, the displacements become:

$$D\varphi_r = \frac{r}{4}(2\log r - 1)\left(-\frac{P_0}{2\pi}\right) + \frac{rc_1}{2}, \tag{87}$$

$$k_s Ghw_0(r) = -\int \frac{1}{r}\int r\, p(r)drdr - \frac{P_0}{2\pi}\log r$$
$$-\frac{k_s Gh}{D}\left(-I(r) - \frac{P_0}{2\pi}\frac{r^2}{4}(\log r - 1) + \frac{r^2}{4}\frac{P_0}{4\pi}(2\log r_0 - 1)\right) + c_3. \tag{88}$$

Enforcement of the boundary conditions Eqs. (61) and (62) in Eqs. (87) and (88) yield the following constants of integration:

$$c_1 = \frac{P_0}{2\pi}(2\log r_0 - 1), \tag{89}$$

$$c_3 = \frac{P_0 \log r_0}{2\pi} + \frac{k_s Gh}{D}\frac{P_0 r_0^2}{16\pi}. \tag{90}$$

Hence the solutions for transverse deflections and rotation functions become:

$$w_0(r) = \frac{P_0 r_0^2}{16D}\left(1 - \left(\frac{r}{r_0}\right)^2 + 2\left(\frac{r}{r_0}\right)^2 \log_e\left(\frac{r}{r_0}\right)\right) - \frac{P_0}{2\pi k_s Gh}\log_e\left(\frac{r}{r_0}\right) = w_0^f(r) + w_0^s(r), \tag{91}$$

$$\varphi(r) = \frac{-P_0 r}{4\pi D}\log_e\left(\frac{r}{r_0}\right), \tag{92}$$

$$w_0(r) = \frac{P_0 r_0^2}{16D}\left(1 - \left(\frac{r}{r_0}\right)^2 + 2\left(\frac{r}{r_0}\right)^2 \log_e\left(\frac{r}{r_0}\right)\right) - \frac{P_0 h^3}{2\pi k_s h 6D(1-\mu)}\log_e\left(\frac{r}{r_0}\right)$$
$$= \frac{P_0 r_0^2}{16\pi D}\left(1 - \left(\frac{r}{r_0}\right)^2 + 2\left(\frac{r}{r_0}\right)^2 \log_e\left(\frac{r}{r_0}\right)\right) - \frac{P_0 h^2}{12\pi D k_s(1-\mu)}\log_e\left(\frac{r}{r_0}\right). \tag{93}$$

4.3. Case 3: First order shear deformable solid circular plate with simply supported edge and under uniformly distributed transverse load

The essential and natural boundary conditions of the simple supports at the edge $r = r_0$ are:

$$w(r = r_0) = 0, \tag{94}$$
$$M_{rr}(r = r_0) = 0. \tag{95}$$

At the centre, $r = 0$:

$$(r\,Q_r)|_{r=0} = 0. \tag{96}$$

From Eq. (93) it is found that:

$$c_0 = 0. \tag{97}$$

For non singular solutions for the displacements, the expression for φ_r gives the condition for bounded solutions as:

$$c_2 = 0. \tag{98}$$

Hence, for simply supported FSDP under uniformly distributed transverse load:

$$k_s Gh w_0(r) = -\int \frac{1}{r} \int r p(r) dr dr - \frac{k_s Gh}{D}\left(-I(r) + \frac{r^2 c_1}{4}\right) + c_3, \tag{99}$$

$$D\varphi_r = -I'(r) + \left(\frac{r}{2}\log r - \frac{r}{4}\right)c_0 + \frac{r}{2}c_1 + \frac{c_2}{r}, \tag{100}$$

$$D\varphi_r = -I'(r) + \frac{r}{2}c_1, \tag{101}$$

$$D\varphi'_r = -I''(r) + \frac{c_1}{2}, \tag{102}$$

$$\varphi'_r = \frac{1}{D}\left(-I''(r) + \frac{c_1}{2}\right), \tag{103}$$

$$M_{rr} = D\left(\frac{\mu}{rD}\left(-I'(r) + \frac{r}{2}c_1\right) + \frac{1}{D}\left(\frac{c_1}{2} - I''(r)\right)\right) = \frac{\mu}{r}\left(\frac{rc_1}{2} - I'(r)\right) + \frac{c_1}{2} - I''(r), \tag{104}$$

$$M_{rr}(r) = \frac{\mu}{r}\left(r\frac{c_1}{2} - \frac{pr^3}{16}\right) + \frac{c_1}{2} - \frac{3p_0 r^2}{16}. \tag{105}$$

The force boundary condition yields:

$$M_{rr}(r = r_0) = \frac{\mu}{r_0}\left(\frac{r_0 c_1}{2} - \frac{p_0 r_0^3}{16}\right) + \frac{c_1}{2} - \frac{3p_0 r_0^2}{16} = 0. \tag{106}$$

Solving:

$$c_1 = \frac{p_0 r_0^2}{8}\left(\frac{3+\mu}{1+\mu}\right). \tag{107}$$

The deflection boundary condition at the simply supported edge gives:

$$w(r = r_0) = -\frac{p_0 r_0^2}{4} + \frac{p_0 r_0^4 k_s Gh}{64D} - \frac{k_s Gh r_0^2}{4D}\frac{p_0 r_0^2}{8}\left(\frac{3+\mu}{1+\mu}\right) + c_3 = 0. \tag{108}$$

Solving:

$$c_3 = \frac{k_s G h}{D} \frac{p_0 r_0^4}{64} \left(\frac{5+\mu}{1+\mu}\right) + \frac{p_0 r_0^2}{4}. \tag{109}$$

Thus, the solutions for the displacements become:

$$w_0(r) = \frac{p_0 r_0^4}{64D}\left(\left(\frac{5+\mu}{1+\mu}\right) - \left(\frac{r}{r_0}\right)^4 - 2\left(\frac{r}{r_0}\right)^2 \left(\frac{3+\mu}{1+\mu}\right)\right) + \frac{p_0 r_0^2}{4k_s G h}\left(1 - \left(\frac{r}{r_0}\right)^2\right) \tag{110}$$

$$= w_0^f(r) + w_0^s(r),$$

$$\varphi(r) = \frac{p_0 r_0^3}{16D}\left(\frac{r}{r_0}\right)\left(\frac{3+\mu}{1+\mu} - \left(\frac{r}{r_0}\right)^2\right). \tag{111}$$

4.3.1. Maximum deflection

The maximum deflection $w_{\max}$ occurs at the plate centre, $r = 0$, and is found as:

$$w_{\max} = w(r = 0) = \frac{p_0 r_0^4}{64D}\left(\frac{5+\mu}{1+\mu}\right) + \frac{p_0 r_0^2}{4k_s G h}, \tag{112}$$

$$w_{\max} = \frac{p_0 r_0^4}{64D}\left(\frac{5+\mu}{1+\mu}\right) + \frac{p_0 r_0^2}{4k_s h G}, \tag{113}$$

$$w_{\max} = \frac{p_0 r_0^4}{D}\left(\left(\frac{5+\mu}{1+\mu}\right)\frac{1}{64} + \frac{1}{24(1-\mu)k_s}\left(\frac{h}{r_0}\right)^2\right). \tag{114}$$

For $\mu = 0.3$, $k_s = 5/6$:

$$w_{\max} = \frac{p_0 r_0^4}{D}\left(\left(\frac{5.3}{1.3}\right)\frac{1}{64} + \frac{1}{24(0.7)\frac{5}{6}}\left(\frac{h}{r_0}\right)^2\right). \tag{115}$$

4.3.2. Maximum rotation

The maximum rotation $(\varphi_{\max})$ occurs at the edge $r = r_0$ is given by:

$$\varphi_{\max} = \varphi_r(r = r_0) = \frac{p_0 r_0^3}{8D(1+\mu)}. \tag{116}$$

4.3.3. Bending moments

The bending moment distribution $M_{rr}(r)$ is given by:

$$M_{rr} = \mu\left(\frac{c_1}{2} - \frac{p_0 r^2}{16}\right) + \frac{c_1}{2} - \frac{3p_0 r^2}{16} = \frac{\mu c_1}{2} - \frac{\mu p_0 r^2}{16} + \frac{c_1}{2} - \frac{3p_0 r^2}{16}, \tag{117}$$

$$M_{rr}(r) = \frac{c_1}{2}(1+\mu) - \frac{p_0 r^2}{16}(3+\mu) = \frac{p_0 r_0^2}{16}(3+\mu) - \frac{p_0 r^2}{16}(3+\mu), \tag{118}$$

$$M_{rr}(r) = \frac{p_0(3+\mu)}{16}(r_0^2 - r^2). \tag{119}$$

The bending moment at the centre of the plate is given by:

$$M_{rr}(r = 0) = \frac{p_0\, r_0^2(3 + \mu)}{16}. \tag{120}$$

5. Validation of the solution for simply supported circular Mindlin plate using the Ritz finite element method

The total potential energy functional Π for a Mindlin's first order shear deformable circular plate is given by:

$$\Pi = \frac{1}{2}\iint_A \int_{-h/2}^{h/2} (\sigma_{rr}\varepsilon_{rr} + \sigma_{\theta\theta}\varepsilon_{\theta\theta} + \tau_{r\theta}\gamma_{r\theta} + \tau_{rz}\gamma_{rz} + \tau_{z\theta}\gamma_{z\theta})dzrdrd\theta - \iint_A pwrdrd\theta, \tag{121}$$

where A is the cross-sectional area (domain) of the circular plate given by:

$$0 \le r \le r_0, \quad 0 \le \theta \le 2\pi.$$

From the principle of virtual displacements:

$$\delta\Pi = 0, \tag{122}$$

$$\iint_A \left[M_{rr}\frac{\partial\delta\varphi_r}{\partial r} + M_{r\theta}\left(\frac{1}{r}\frac{\partial\delta\varphi_r}{\partial\theta} + \frac{\partial\delta\varphi_\theta}{\partial r} - \frac{\partial\varphi_\theta}{r}\right) + M_{\theta\theta}\left(\frac{\delta\varphi_r}{r} + \frac{1}{r}\frac{\partial\delta\varphi_\theta}{\partial\theta}\right)\right.$$

$$\left. + Q_r\left(\delta\varphi_r + \frac{\partial\delta w_0}{\partial r}\right) + Q_\theta\left(\delta\varphi_\theta + \frac{1}{r}\frac{\partial\delta w_0}{\partial\theta}\right) - p\delta w_0\right]rdrd\theta = 0, \tag{123}$$

where:

$$Q_r = k_s\int_{-h/2}^{h/2} \tau_{z\theta}\,dz, \tag{124}$$

$$Q_r = k_s\int_{-h/2}^{h/2} \tau_{zr}\,dz. \tag{125}$$

The weak form of the virtual work statement is:

$$\int_0^{2\pi}\int_0^{r_0}\left[D\left(\frac{d}{dr}\varphi_r(r) + \mu\frac{\varphi_r(r)}{r}\right)\frac{d\delta\varphi_r(r)}{dr} + D\left(\mu\frac{d\varphi_r(r)}{dr} + \frac{\varphi_r(r)}{r}\right)\frac{\delta\varphi_r(r)}{r}\right.$$

$$\left. + k_sGh\left(\varphi_r(r) + \frac{dw_0(r)}{dr}\right)\left(\delta\varphi_r(r) + \frac{d\delta w_0(r)}{dr}\right) - p(r)\delta w_0(r)\right]rdrd\theta = 0, \tag{126}$$

$$2\pi\int_0^{r_0}\left[D\left(\frac{d\varphi_r(r)}{dr} + \mu\frac{\varphi_r(r)}{r}\right)\frac{d\delta\varphi_r}{dr} + D\left(\mu\frac{d\varphi_r(r)}{dr} + \frac{\varphi_r(r)}{r}\right)\frac{\delta\varphi_r(r)}{r}\right.$$

$$\left. + k_sGh\left(\varphi_r(r) + \frac{dw_0(r)}{dr}\right)\left(\delta\varphi_r(r) + \frac{d\delta w_0(r)}{dr}\right) - p(r)\delta w_0(r)\right]rdr = 0, \tag{127}$$

$$\int_0^{r_0}\left[D\left(\frac{d\varphi_r}{dr} + \mu\frac{\varphi_r}{r}\right)\delta\frac{d\varphi_r}{dr} + D\left(\mu\frac{d\varphi_r}{dr} + \frac{\varphi_r}{r}\right)\frac{\delta\varphi_r}{r} + k_sGh\left(\varphi_r + \frac{dw_0}{dr}\right)\right.$$

$$\left. \times\left(\delta\varphi_r + \delta\frac{dw_0(r)}{dr}\right) - p(r)\delta w_0(r)\right]rdr = 0. \tag{128}$$

Let:

$$w_0(r) = \sum_{i=1}^{M} a_i f_i(r), \tag{129}$$

$$\varphi_r(r) = \sum_{j=1}^{N} c_j g_j(r). \tag{130}$$

Then:

$$\frac{d\varphi_r(r)}{dr} = \sum_{j=1}^{N} c_j \frac{dg_j(r)}{dr}, \tag{131}$$

$$\frac{dw_0(r)}{dr} = \sum_{i=1}^{M} a_i \frac{df_i(r)}{dr}, \tag{132}$$

$$\int_0^{r_0} \left[D\left(\sum c_j \frac{dg_j}{dr} + \mu \sum c_j g_j \right) \delta \sum c_k \frac{dg_k}{dr} \right.$$
$$+ D\left(\mu \sum c_j \frac{dg_j}{dr} + \frac{1}{r} \sum c_j g_j(r) \right) \frac{1}{r} \delta \sum c_k g_k \tag{133}$$
$$\left. + k_s Gh \left(\sum c_j g_j + \sum a_i \frac{df_i}{dr} \right) \left(\delta \sum c_k g_k + \delta \sum a_k \frac{df_k}{dr} \right) - p(r) \delta \sum a_i f_i \right] r dr = 0.$$

Equating the coefficients of δa_i and δc_i to zero, and this corresponds to the requirement that $\partial \Pi = 0$ with respect to a_i and $\partial \Pi = 0$ with respect to c_i we have the following Ritz variational statements:

$$\sum_{j=1}^{M} \left[k_s Gh \int_0^{r_0} \frac{df_i(r)}{dr} \frac{df_j(r)}{dr} r dr \right] a_j + \sum_{j=1}^{M} \left[k_s Gh \int_0^{r_0} \frac{df_i}{dr} g_j(r) r dr \right] c_j$$
$$- \int_0^{r_0} p(r) f_i(r) r \, dr = 0, \tag{134}$$

$$k_s Gh \sum_{j=1}^{M} \left[\int_0^{r_0} g_k(r) \frac{df_j(r)}{dr} r \, dr \right] a_j + \sum_{j=1}^{N} \left\{ k_s Gh \int_0^{r_0} g_k(r) g_j(r) r dr \right.$$
$$\left. + D \int_0^{r_0} \left[\frac{dg_k(r)}{dr} \frac{dg_j(r)}{dr} + \frac{\mu}{r} \left(g_k(r) \frac{dg_j(r)}{dr} + \frac{dg_k}{dr} g_j(r) \right) + \frac{1}{r^2} g_k(r) g_j(r) \right] r dr \right\} c_j = 0, \tag{135}$$

where $k = 1, 2, \ldots, N$.

In matrix form, the Ritz variational equations are:

$$\begin{bmatrix} [X_{ij}] & [Y_{ij}] \\ [Y_{ij}]^T & [Z_{ij}] \end{bmatrix} \begin{pmatrix} [a_i] \\ [c_j] \end{pmatrix} = \begin{Bmatrix} P_i \\ 0 \end{Bmatrix}, \tag{136}$$

in which:

$$[X_{ij}] = k_s Gh \int_0^{r_0} \frac{df_i(r)}{dr} \frac{df_j(r)}{dr} r dr, \tag{137}$$

$$[X_{ij}] = k_s Gh \int_0^{r_0} f_i'(r) f_j'(r) r dr, \tag{138}$$

$$[Y_{ij}] = k_s Gh \int_0^{r_0} \frac{df_i(r)}{dr} g_j(r) r dr, \tag{139}$$

$$[Y_{ij}] = k_s Gh \int_0^{r_0} f_i'(r) g_j(r) r dr, \tag{140}$$

$$[P_i] = \int_0^{r_0} p(r) f_i(r) r dr, \tag{141}$$

$$[Z_{kj}] = D \int_0^{r_0} \left[\frac{dg_k(r)}{dr} \frac{dg_j(r)}{dr} + \frac{\mu}{r} \left(g_k(r) \frac{dg_j(r)}{dr} + \frac{dg_k(r)}{dr} g_j(r) + \frac{1}{r^2} g_k(r) g_j(r) \right) \right] r dr$$
$$+ k_s Gh \int_0^{r_0} g_k(r) g_j(r) r dr. \tag{142}$$

5.1.1. Simply supported solid first order shear deformable circular plate

Shape functions for simply supported solid first order shear deformable circular plates are:

$$f_1(r) = 1 - \frac{r}{r_0}, \tag{143}$$

$$f_2(r) = 1 - \frac{r^2}{r_0^2}, \tag{144}$$

$$g_1(r) = \frac{r}{r_0}, \tag{145}$$

where f_1, f_2 and g_1 satisfy the essential boundary conditions:

$$w_0(r = r_0) = 0, \tag{146}$$
$$\varphi_r(r = 0) = 0, \tag{147}$$
$$X_{11} = k_s Gh \int_0^{r_0} f_1'(r) f_1'(r) r dr, \tag{148}$$

$$f_1'(r) = \left(-\frac{1}{r_0} \right), \tag{149}$$

$$X_{11} = k_s Gh \int_0^{r_0} \left(-\frac{1}{r_0} \right) \left(-\frac{1}{r_0} \right) r dr = \frac{k_s Gh}{r_0^2} \int_0^{r_0} r dr =$$
$$X_{11} = \frac{k_s Gh}{r_0^2} \left[\frac{r^2}{2} \right]_0^{r_0} = \frac{k_s Gh}{r_0^2} \frac{r_0^2}{2} = \frac{k_s Gh}{2}, \tag{150}$$

$$X_{12} = k_s Gh \int_0^{r_0} f_1'(r) f_2'(r) r \, dr, \tag{151}$$

$$X_{12} = k_s Gh \int_0^{r_0} \left(-\frac{1}{r_0} \right) \left(-\frac{2r}{r_0^2} \right) r dr = \frac{2k_s Gh}{r_0^3} \int_0^{r_0} r^2 dr, \tag{152}$$

$$X_{12} = \frac{2k_s Gh}{r_0^3} \left[\frac{r^3}{3} \right]_0^{r_0} = \frac{2k_s Gh}{r_0^3} \frac{r_0^3}{3} = \frac{2k_s Gh}{3}, \tag{153}$$

$$Y_{11} = k_s Gh \int_0^{r_0} \left(-\frac{1}{r_0} \right) \left(\frac{r}{r_0} \right) r dr, \tag{154}$$

$$Y_{11} = -\frac{k_s Gh}{r_0^2} \int_0^{r_0} r^2 dr = -\frac{k_s Gh}{r_0^2} \left[\frac{r^3}{3} \right]_0^{r_0} = -\frac{k_s Gh}{r_0^2} \frac{r_0^3}{3} = -\frac{k_s Gh r_0}{3}, \tag{155}$$

$$Y_{21} = k_s Gh \int f_2' g_1 r dr, \tag{156}$$

$$Y_{21} = k_s Gh \int_0^{r_0} \left(-\frac{2r}{r_0^2} \right) \frac{r}{r_0} r \, dr, \tag{157}$$

$$Y_{21} = -\frac{2k_s Gh}{r_0^3} \int_0^{r_0} r^3 \, dr = -\frac{2k_s Gh}{r_0^3} \left[\frac{r^4}{4} \right]_0^{r_0} = -\frac{2k_s Gh}{r_0^3} \left(\frac{r_0^4}{4} \right) = -\frac{k_s Gh r_0}{2}, \tag{158}$$

$$X_{21} = k_s Gh \int_0^{r_0} f_2'(r) f_1' \, r \, dr, \tag{159}$$

$$X_{21} = k_s Gh \int_0^{r_0} \left(-\frac{2r}{r_0^2} \right) \left(-\frac{1}{r_0} \right) r \, dr = \frac{2k_s Gh}{r_0^3} \int_0^{r_0} r^2 \, dr = \frac{2k_s Gh}{r_0^3} \left[\frac{r^3}{3} \right]_0^{r_0}, \tag{160}$$

$$X_{21} = \frac{2k_s Gh}{r_0^3} \left(\frac{r_0^3}{3} \right) = \frac{2}{3} k_s Gh, \tag{161}$$

$$X_{22} = k_s Gh \int_0^{r_0} f_2'(r) f_2'(r) \, r \, dr, \tag{162}$$

$$X_{22} = k_s Gh \int_0^{r_0} \left(-\frac{2r}{r_0^2} \right) \left(-\frac{2r}{r_0^2} \right) r \, dr, \tag{163}$$

$$X_{22} = \frac{4k_s Gh}{r_0^4} \int_0^{r_0} r^3 \, dr = \frac{4k_s Gh}{r_0^4} \left(\frac{r^4}{4} \right)_0^{r_0} = \frac{4k_s Gh}{r_0^4} \frac{r_0^4}{4} = k_s Gh, \tag{164}$$

$$Z = D \int_0^{r_0} \left(\left(\frac{1}{r_0} \right) \left(\frac{1}{r_0} \right) + \frac{\mu}{r} \left(\frac{r}{r_0} \frac{1}{r_0} + \frac{1}{r_0} \frac{r}{r_0} \right) + \frac{1}{r^2} \frac{r}{r_0} \frac{r}{r_0} \right) r \, dr + k_s Gh \int_0^{r_0} \left(\frac{r}{r_0} \right) \left(\frac{r}{r_0} \right) r \, dr, \tag{165}$$

$$Z = D \int_0^{r_0} \left(\frac{1}{r_0^2} + \frac{\mu}{r} \left(\frac{2r}{r_0^2} \right) + \frac{1}{r_0^2} \right) r \, dr + \frac{k_s Gh}{r_0^2} \int_0^{r_0} r^3 \, dr, \tag{166}$$

$$Z = \frac{D}{r_0^2} \int_0^{r_0} (2 + 2\mu) r \, dr + \frac{k_s Gh}{r_0^2} \int_0^{r_0} r^3 \, dr, \tag{167}$$

$$Z = \frac{(2 + 2\mu)D}{r_0^2} \int_0^{r_0} r \, dr + \frac{k_s Gh}{r_0^2} \int_0^{r_0} r^3 \, dr, \tag{168}$$

$$Z = \frac{(2 + 2\mu)D}{r_0^2} \left[\frac{r^2}{2} \right]_0^{r_0} + \frac{k_s Gh}{r_0^2} \left[\frac{r^4}{4} \right]_0^{r_0}, \tag{169}$$

$$Z = (1 + \mu)D + \frac{k_s Gh r_0^4}{r_0^2 \, 4} = (1 + \mu)D + \frac{k_s Gh}{4} r_0^2, \tag{170}$$

$$Z = \frac{k_s Gh}{12} \left(3r_0^2 + \frac{12(1 + \mu)D}{k_s Gh} \right), \tag{171}$$

$$P_1 = \int_0^{r_0} p_0 \left(1 - \frac{r}{r_0} \right) r \, dr = p_0 \int_0^{r_0} \left(1 - \frac{r}{r_0} \right) r \, dr = p_0 \int_0^{r_0} \left(r - \frac{r^2}{r_0} \right) dr, \tag{172}$$

$$P_1 = p_0 \left[\frac{r^2}{2} - \frac{r^3}{3r_0} \right]_0^{r_0} = p_0 \left(\frac{r_0^2}{2} - \frac{r_0^3}{3r_0} \right), \tag{173}$$

$$P_1 = p_0 \left(\frac{r_0^2}{2} - \frac{r_0^2}{3} \right) = \frac{p_0 r_0^2}{6}, \tag{174}$$

$$P_2 = \int_0^{r_0} p_0 \left(1 - \frac{r^2}{r_0^2} \right) r \, dr = p_0 \int_0^{r_0} \left(r - \frac{r^3}{r_0^2} \right) dr, \tag{175}$$

$$P_2 = p_0 \left[\frac{r^2}{2} - \frac{r^4}{4r_0^2} \right]_0^{r_0} = p_0 \left(\frac{r_0^2}{2} - \frac{r_0^2}{4} \right) = \frac{p_0 r_0^2}{4}. \tag{176}$$

The Ritz variational equations become in matrix form:

$$\frac{k_s Gh}{12}\begin{bmatrix} 6 & 8 & -4r_0 \\ 8 & 12 & -6r_0 \\ -4r_0 & -6r_0 & \left(3r_0^2 + \dfrac{12(1+\mu)D}{k_s Gh}\right) \end{bmatrix}\begin{pmatrix} a_1 \\ a_2 \\ c_1 \end{pmatrix} = \frac{p_0 r_0^2}{12}\begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix}, \tag{177}$$

$$\begin{pmatrix} 6 & 8 & -4r_0 \\ 8 & 12 & -6r_0 \\ -4r_0 & -6r_0 & \left(3r_0^2 + \dfrac{12(1+\mu)D}{k_s Gh}\right) \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \\ c_1 \end{pmatrix} = \frac{p_0 r_0^2}{k_s Gh}\begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix}. \tag{178}$$

The undetermined displacement parameters are:

$$a_1 = 0, \tag{179}$$

$$a_2 = \frac{p_0 r_0^2}{24}\left(\frac{4}{k_s Gh} + \frac{r_0^2}{D(1+\mu)}\right), \tag{180}$$

$$c_1 = \frac{2 p_0 r_0^3}{3D(1+\mu)}. \tag{181}$$

The Ritz variational solution is:

$$w_2(r) = \frac{p_0 r_0^4}{24D(1+\mu)}\left(1 - \frac{r^2}{r_0^2}\right) + \frac{p_0 r_0^2}{6 k_s Gh}\left(1 - \frac{r^2}{r_0^2}\right), \tag{182}$$

$$\varphi_1(r) = \frac{2 p_0 r_0^3}{3D(1+\mu)}\left(\frac{r}{r_0}\right). \tag{183}$$

The Ritz variational solutions are only approximate.

5.2. Validation of solution for clamped circular Mindlin plate

The solutions obtained in this study for Mindlin's first order shear deformable solid circular plate with clamped edge and under uniform transverse load distribution are exactly the same as solutions obtained by Ike et al. [31] who used the method of direct integration of the biharmonic equation for circular Mindlin plates, and the enforcement of appropriate boundary conditions.

6. Discussion

In this work, the governing partial differential equations of equilibrium for axially symmetric flexural problems of first order shear deformable circular plates were solved analytically to obtain closed form mathematical solutions for the unknown generalised displacements $w_0(r)$ and $\varphi_r(r)$. The governing equations, Eqs. (40) and (41), for axially symmetric first order shear deformable (FSD) circular plate problems were first expressed in terms of the unknown generalised displacements with the aid of the differential equations relating the stress resultants and the generalised displacements Eqs. (42-44). The resulting differential expressions, now in variable separable form were integrated to obtain the general solutions for the unknown displacements as Eqs. (50) and (58). The general solution for the rotation function $\varphi_r(r)$ contained three unknown constants of integration; while the general solutions for the deflection $w_0(r)$ contained four unknown constants of integration. Mathematical solutions for $w_0(r)$ and $\varphi_r(r)$ were then sought for three specific cases.

For first order shear deformable solid circular plate with clamped edge and under uniformly distributed transverse load over the entire plate domain, boundedness conditions for the rotation, and the need for the shear force at the centre to vanish were used to find two constants of integration as Eqs. (59) and (60). The boundary conditions at the clamped edge – Eqs. (61) and

(62) – were used to obtain the other two constants of integration as Eqs. (68) and (69). The deflection function was thus obtained as Eq. (70) and the rotation function as Eq. (72). It was observed that the deflection function could be decomposed into a flexural component, given by Eq. (74) and a shear component given by Eq. (75). The maximum deflection was found to occur at the centre of the plate, in line with the demands of the symmetry of the plate and the loading. The maximum deflection was found as Eqs. (76) or (77). For the particular case where $k_s = 5/6$ and $\mu = 0.30$, the maximum deflection was obtained as Eq. (78). The maximum deflection of first order shear deformable circular plate with clamped edge for the case of uniformly distributed transverse load are shown in Table 1 for various values of the ratio of the plate thickness to the radius (h/r_0). The difference between the maximum deflection values obtained using the first order shear deformable circular plate with clamped edge for the case of uniformly distributed load were calculated for varying ratios of h/r_0 and shown in Table 2. An examination of Table 2 shows that the contribution of shear deformation to the overall bending deformation of the plate increases as the plate thickness increases. For thin plates, the difference between the maximum transverse deflection obtained using the first order shear deformable circular plate and the classical Kirchhoff-Love circular thin plate theory is insignificant. However, as the plate thickness increases, the difference becomes significant.

For the case of first order shear deformable solid circular plate with clamped edge and point load P_0 applied at the centre, the mathematical solutions for $w_0(r)$ and $\varphi_r(r)$ were obtained as Eqs. (91) and (92), respectively. The deflection $w_0(r)$ in the first order shear deformable circular plate was found to be decomposed into two components, flexural component, and shear component, where the flexural component corresponded to the solution for the Kirchhoff-Love circular thin plate.

It is also observed that:

$$\varphi(r) = \frac{-dw_0^f(r)}{dr}. \tag{184}$$

It is observed that the shear component of the deflection is singular and undefined at $r = 0$, the point of application of the point load P_0. It is thus mathematically impossible to determine the maximum deflection since the deflection is singular at the centre of the plate where $r = 0$.

For the case of first order shear deformable solid circular plate with simply supported edge and under uniformly distributed transverse load over the entire plate region, the general solutions for $w_0(r)$ and $\varphi_r(r)$ were obtained as Eqs. (110) and (111) respectively. The deflection $w_0(r)$ was observed to be decomposed into two components; a flexural component and shear component. The maximum deflection was found to occur at the plate centre, in line with the symmetrical nature of the problem. The maximum deflection was obtained as Eqs. (113) or (114). The maximum deflection was also found in terms of the ratio h/r_0 as Eq. (115). The maximum rotation was found to occur at the plate edge and was obtained as Equation (116). The bending moment distribution $M_{rr}(r)$ was obtained as Eq. (119). Maximum bending moment M_{rrmax} was found to occur at the centre of the plate and was obtained as Eq. (120).

The significant contributions of the present study are:

(i) Closed form mathematical solutions were obtained for the general case of Mindlin's first order shear deformable circular plate subject to an arbitrary distribution of transverse load as Eqs. (50) and (58). The solutions were derived from first principles, and were derived to satisfy the governing equations of equilibrium at all points on the solution domain, which is the two dimensional domain of the circular plate $0 \le r \le r_0$, $0 \le \theta \le 2\pi$.

(ii) Mathematical solutions were derived from the general solution for the particular cases of clamped edge and uniformly distributed load as Eqs. (70) and (72). The solutions were found to satisfy the boundary conditions and were derived such that all points on the circular plate domain satisfied the governing equations. For point load at the centre, the mathematical solutions were

found from the general solution as Eqs. (92) and (93).

(iii) Mathematical solutions were similarly derived from the general solution for the particular case of simply supported edge and uniformly distributed load as Eqs. (110) and (111).

7. Conclusions

The following conclusions are made from the study:

(i) The partial differential equations of equilibrium of axisymmetrically loaded first order shear deformable circular plates have been solved mathematically using the method of separation of variables.

(ii) The general solutions obtained for the transverse deflection $w_0(r)$ and the rotation functions for any arbitrary axisymmetric load distribution contained four constants of integration, which are determined for specific cases of edge supports and transverse loads by the application of the appropriate boundary conditions.

(iii) The general solution obtained for the deflection of first order shear deformable solid circular plates fixed at the edge $r = r_0$ was decomposable into the flexural component and a shear component. The flexural component of the deflection in this case was exactly the same as the deflection expression for a Kirchhoff-Love circular plate of the same radius. The contribution of the shear strength component to the bending deformation increased significantly with increase in the ratio of plate thickness to the radius (h/r_0).

(iv) The shear force distribution for first order shear deformable solid circular plates vanished when:

$$\varphi_r = \frac{dw_0(r)}{dr}. \tag{185}$$

(v) The deflections obtained for first order shear deformable solid circular plates with clamped edge ($r = r_0$) and subject to a point load P_0 applied at the centre was decomposable into a flexural component and a shear component. The deflections in this case also increased significantly with significant increase in the ratio of the plate thickness to the radius (h/r_0).

(vi) For first order shear deformable solid circular plates with simply supported edge ($r = r_0$) and subject to uniformly distributed transverse load, the solution obtained for deflection was decomposed into a flexural component and a shear component. The deflection increased significantly with increase in the ratio of the plate thickness to the radius (h/r_0). The flexural component of the bending deflection was found to be exactly identical with the deflections obtained using the Kirchhoff-Love theory for the circular plate with simply supported edge ($r = r_0$).

(vii) For all the cases considered the maximum deflection was found to occur at the plate centre; in line with the requirements of symmetry of the problems. The expressions for the maximum deflection in all the three cases were decomposable into flexural component and shear component.

(viii) For first order shear deformable solid circular plates with simply supported edge ($r = r_0$) the maximum rotation was found to occur at the simply supported edge ($r = r_0$).

(ix) Mathematical (closed form) expressions were obtained for the generalised displacements $w_0(r)$ and $\varphi_r(r)$ for any given load distribution $p(r)$ and any support (restraint) condition.

(x) Mathematical (closed form) expressions were obtained for the unknown generalised displacements in the boundary value problem for specific cases of fixed edge and simply supported edge and for uniformly distributed transverse load on the plate domain and for point load applied at the centre.

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