Mathematical modeling of three-layer beam hydroelastic oscillations

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Abstract. The problem of hydroelastic oscillations of three-layer beam interacting with viscous liquid layer is set up and analytically solved. The problem presents the equation system of a three-layer beam and Navier-Stokes equations. The following boundary conditions are chosen: the no-slip conditions, the conditions for pressure at the edges, the simply supported edges conditions. The problem is solved for the steady-state harmonic regime. The frequency dependent distribution functions of the beam deflection are constructed. The given function allows investigating the resonance hydroelastic oscillations of a three-layer beam, as well as its deflected mode.

Keywords: hydroelasticity, vibration, three-layer beam, viscous liquid, mathematical modeling.

1. Introduction

The studies of elastic construction elements oscillation with consideration of their interaction with a liquid present an important issue of contemporary technology. For instance, the bending oscillations of a beam under interaction with an ideal incompressible liquid were considered in [1]. This solution made it possible to investigate the cylinder cavitations resource of the combustion engine with water cooling. The chaotic oscillations of the plate, interacting with an ideal incompressible liquid flow, are studied in [2]. Reference [3] is devoted to numerical study of rectangular plate hydroelasic oscillations, the plate being fully plunged into motionless ideal liquid or floating on its free surface. The analogous study for the case of rectangular plates interacting with the ideal liquid flow is made in [4]. However, the liquid viscosity is excluded from consideration in the above mentioned studies, but this property is very important, because it determines damping characteristics of oscillation system.

The study of vibrations of infinite length beam on viscous liquid layer was made in [5]. The study of hydroelastic oscillations of the elastic-fixed rigid wall of the finite sizes narrow channel is carried out in [6]. Hydroelastic oscillations of cantilever beam, plunged into a viscous incompressible liquid, are investigated in [7]. The problem of vibrating circular plates interacting with a layer of viscous incompressible liquid between them is solved in [8]. The analogous plane problem for the two vibrating plates of finite sizes is studied in [9]. Reference [10] investigates hydroelastic oscillations of the beam in a viscous liquid flow for the case of piezo-electric elements with the aim of obtaining energy from the flow. The problem of the bending hydroelastic oscillations of the plate under the impact of the viscous liquid pulsating layer was studied in [11]. The study of the hydroelastic oscillations of the plate, resting on Winkler foundation was made in [12-14].

References [15-18] consider the statics and dynamics of three-layer beams and plates under the local and distributed loads of various natures. The investigation of elastic three-layer plate dynamic interaction with a viscous liquid layer is of theoretical importance, while its results are of practical interest for computing and analyzing the new technology objects. But there is a shortage of studies devoted to the interaction of three-layer beams and plates with viscous liquid. For instance, the forced hydroelastic oscillations of the three-layered circular plate, interacting

with viscous incompressible liquid layer under the channel foundation vibration, are investigated in [19]. Alongside with this, the hydroelastic oscillations of three-layer beam are of theoretical and practical interest, too.

2. Statement of the problem

Let us consider the three-layer beam interacting with viscous incompressible liquid layer in conditions of foundation vibration (Fig. 1). We will study the plane hydroelastic problem. The foundation vibration takes place only in a vertical plane. The three-layer beam length b is considerably larger than its width 2ℓ. The liquid pressure at the right and left edges is constant p_0, and liquid leakage may be considered to be free. The liquid layer thickness is $h_0 \ll \ell$. The three-layer beam is the package consisting of bearing layers 1, 2, their thickness being h_1 and h_2, and incompressible lightweight filler 3 with the thickness $2c$. The zig-zag hypothesis is valid for the three-layer beam, i.e. Kirchhoff hypothesis is valid for bearing layers, as well as the normal in the beam filler remains straight and turning by the angle φ [13-17]. Let us connect Cartesian coordinate Oxz with the center of beam filler medium surface in an undisturbed state. The rigid diaphragms, hindering the relative layers shift, but not impeding the deformation from its plane, are supposed to be situated at the beam edges [15-17]. Hence, the three-layer beam oscillations are caused by foundation vibration, while deformations of the plate are considered to be small. The three-layer beam is simply supported at its edges. Consideration of liquid viscosity leads to quick going down of transition processes, and according to [20], we can exclude the initial processes impact from the very beginning. That is why we will consider the steady-state harmonic oscillations.
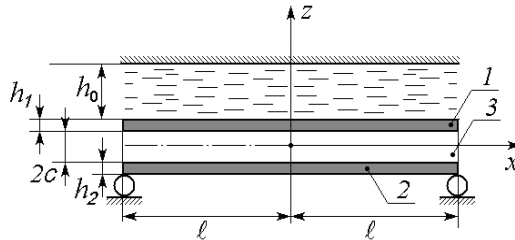

Fig. 1. A schematic diagram of the three-layer beam interacting with thin liquid layer

3. The theory

The law of foundation vibration motion will be presented in the form of:

$$z_0 = E_z f_0(\omega t), \quad f_0(\omega t) = \sin(\omega t). \tag{1}$$

Then acceleration of vibration foundation can be written as:

$$\ddot{z}_0 = E_z \frac{d^2 f_0(\omega t)}{dt^2} = -E_z \omega^2 f_0(\omega t), \tag{2}$$

where E_z is the amplitude of foundation oscillation, ω is the frequency, t is the time.

The longitudinal waves are not studied, i.e. inertial forces in a longitudinal direction are not considered according to [21], while we study bending oscillations of the three-layer beam. In this case, dynamics equations of three-layer beam with incompressible lightweight filler will be written as [15]:

$$a_1 \frac{\partial^2 u}{\partial x^2} + a_6 \frac{\partial^2 \varphi}{\partial x^2} - a_7 \frac{\partial^3 w}{\partial x^3} = q_{zx}, \quad a_6 \frac{\partial^2 u}{\partial x^2} + a_2 \frac{\partial^2 \varphi}{\partial x^2} - a_3 \frac{\partial^3 w}{\partial x^3} = 0, \tag{3}$$

$$a_7 \frac{\partial^3 u}{\partial x^3} + a_3 \frac{\partial^3 \varphi}{\partial x^3} - a_4 \frac{\partial^4 w}{\partial x^4} - m_0 \left(\frac{\partial^2 w}{\partial t^2} + \ddot{z}_0 \right) = -q_{zz},$$

$$a_1 = K_1^+ h_1 + K_2^+ h_2 + 2 K_3^+ c, \quad a_2 = c^2 \left[K_1^+ h_1 + K_2^+ h_2 + \frac{2}{3} K_3^+ c \right],$$

$$a_3 = c \left[K_1^+ h_1 \left(c + \frac{1}{2} h_1 \right) + K_2^+ h_2 \left(c + \frac{1}{2} h_2 \right) + \frac{2}{3} K_3^+ c^2 \right],$$

$$a_4 = K_1^+ h_1 \left(c^2 + c h_1 + \frac{1}{3} h_1^2 \right) + K_2^+ h_2 \left(c^2 + c h_2 + \frac{1}{3} h_2^2 \right) + \frac{2}{3} K_3^+ c^3,$$

$$a_6 = c (K_1^+ h_1 - K_2^+ h_2), \quad a_7 = K_1^+ h_1 \left(c + \frac{1}{2} h_1 \right) - K_2^+ h_2 \left(c + \frac{1}{2} h_2 \right),$$

$$K_j^+ = K_j + \frac{4}{3} G_j, \quad m_0 = \rho_1 h_1 + \rho_2 h_2 + 2\rho_3 c,$$

where G_j is the shear modulus of jth layer, K_j is the bulk modulus of jth layer, ρ_j is the density of jth layer material, $j = 1, 2, 3$ – layer number, u is the longitudinal displacement of the beam, w is the beam deflection, φ is the angle of rotation of the deformed normal in the beam incompressible lightweight filler, q_{zz} is the normal stress in the viscous liquid layer, q_{zx} is the shear stress in the viscous liquid layer.

The expressions for q_{zz} and q_{zx} take the form of [19, 22]:

$$q_{zz} = -p + 2\rho\nu \frac{\partial u_z}{\partial z} \text{ at } z = w + c + h_1, \quad q_{zx} = \rho\nu \left(\frac{\partial u_z}{\partial x} + \frac{\partial u_x}{\partial z} \right) \text{ at } z = w + c + h_1. \qquad (4)$$

The boundary conditions of Eq. (3) present simply supported edges conditions, i.e.:

$$u = \varphi = w = \frac{\partial^2 w}{\partial x^2} = 0, \text{ at } x = \pm \ell. \qquad (5)$$

The movement of thin viscous liquid layer can be considered as a creeping one [19]. Consequently, the dynamic equations of the liquid represent the Navier-Stokes equation without the inertia members, i.e. [22]:

$$\frac{1}{\rho} \frac{\partial p}{\partial x} = \nu \left(\frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_x}{\partial z^2} \right), \quad \frac{1}{\rho} \frac{\partial p}{\partial z} = \nu \left(\frac{\partial^2 u_z}{\partial x^2} + \frac{\partial^2 u_z}{\partial z^2} \right), \quad \frac{\partial u_x}{\partial x} + \frac{\partial u_z}{\partial z} = 0, \qquad (6)$$

where u_x, u_z are liquid velocity projections on the coordinate axis, ρ is the density of the liquid, ν is the kinematic coefficient of the liquid viscosity, p is the pressure.

The boundary conditions of Eq. (6) are the no-slip conditions and the ones for the pressure at the edges:

$$u_x = 0, \quad u_z = 0, \text{ at } z = h_0 + c + h_1, \quad u_x = \frac{\partial u}{\partial t}, \quad u_z = \frac{\partial w}{\partial t}, \text{ at } z = w + c + h_1, \qquad (7)$$

$$p = p_0 - \rho \ddot{z}_0 (z - h_0 - h_1 - c), \text{ at } x = \pm \ell. \qquad (8)$$

We introduce dimensionless variables and small parameters into the consideration of:

$$\zeta = \frac{(z - c - h_1)}{h_0}, \quad \xi = \frac{x}{l}, \quad \tau = \omega t, \quad w = w_m W, \quad u = u_m U, \quad \varphi = \varphi_m \Phi,$$

$$p = p_0 + \rho\nu \, w_m \omega (h_0 \psi^2)^{-1} P - \rho \ddot{z}_0 (z - h_0 - h_1 - c), \quad u_z = w_m \omega U_\zeta, \qquad (9)$$

$$u_x = \frac{w_m \omega}{\psi} U_\xi, \quad \lambda = \frac{w_m}{h_0} \ll 1, \quad \psi = \frac{h_0}{l} \ll 1.$$

Taking into account Eq. (9), the Eqs. (6)-(8) in dimensionless variables in zero approximation on ψ and λ will be written down, as:

$$\frac{\partial P}{\partial \xi} = \frac{\partial^2 U_\xi}{\partial \zeta^2}, \quad \frac{\partial P}{\partial \zeta} = 0, \quad \frac{\partial U_\xi}{\partial \xi} + \frac{\partial U_\zeta}{\partial \zeta} = 0, \tag{10}$$

$$U_\xi = 0, \quad U_\zeta = 0, \text{ at } \zeta = 1, \quad U_\xi = 0, \quad U_\zeta = \frac{\partial W}{\partial \tau}, \text{ at } \zeta = 0, \quad P = 0, \text{ at } \xi = \pm 1. \tag{11}$$

Further, solving the problem of Eqs. (10, 11) we get:

$$U_\xi = \frac{\zeta^2 - \zeta}{2} \frac{\partial P}{\partial \xi}, \quad U_\zeta = \frac{\partial W}{\partial \tau} + (2\zeta^3 - 3\zeta^2)\frac{\partial W}{\partial \tau}, \tag{12}$$

$$P = -12 \int_{-1}^{\xi} \int \frac{\partial W}{\partial \tau} d\xi d\xi + 6(\xi + 1) \int_{-1}^{1} \int \frac{\partial W}{\partial \tau} d\xi d\xi. \tag{13}$$

The normal and shear stresses Eqs. (4), (5) in variables Eq. (9) take the form of:

$$q_{zz} = -p_0 + \rho h_0 \ddot{z}_0 (\zeta - 1) - \rho v \, w_m \, \omega (h_0 \, \psi^2)^{-1} P, \text{ at } \zeta = 0, \tag{14}$$

$$q_{zx} = \rho v \, w_m \, \omega (h_0 \, \psi)^{-1} \frac{\partial U_\xi}{\partial \zeta}, \text{ at } \zeta = 0. \tag{15}$$

According to the expressions Eqs. (14) and (15) the condition $q_{zz} \gg q_{zx}$ is made for a thin liquid layer. Thus, in comparison with normal stress, the shear one can be neglected, and by substituting Eq. (14) in Eq. (3) we get:

$$a_1 \frac{\partial^2 u}{\partial x^2} + a_6 \frac{\partial^2 \varphi}{\partial x^2} - a_7 \frac{\partial^3 w}{\partial x^3} = 0, \quad a_6 \frac{\partial^2 u}{\partial x^2} + a_2 \frac{\partial^2 \varphi}{\partial x^2} - a_3 \frac{\partial^3 w}{\partial x^3} = 0,$$

$$a_7 \frac{\partial^3 u}{\partial x^3} + a_3 \frac{\partial^3 \varphi}{\partial x^3} - a_4 \frac{\partial^4 w}{\partial x^4} - m_0 \left(\frac{\partial^2 w}{\partial t^2} + \ddot{z}_0 \right) = p_0 + \rho h_0 \ddot{z}_0 + \frac{\rho v \, w_m \, \omega}{h_0 \, \psi^2} P. \tag{16}$$

By using the first and the second equations (16), we find that:

$$\frac{\partial^2 u}{\partial x^2} = b_1 \frac{\partial^3 w}{\partial x^3}, \quad \frac{\partial^2 \varphi}{\partial x^2} = b_2 \frac{\partial^3 w}{\partial x^3}, \quad b_1 = \frac{(a_2 \, a_7 - a_3 a_6)}{(a_1 \, a_2 - a_6^2)}, \quad b_2 = \frac{(a_1 \, a_3 - a_7 \, a_6)}{(a_1 \, a_2 - a_6^2)}. \tag{17}$$

Taking into account Eq. (18), we get from Eq. (17) the equation of a three-layer beam in the form of:

$$D^* \frac{\partial^4 w}{\partial x^4} + m_0 \omega^2 \left(\frac{\partial^2 w}{\partial t^2} + \frac{\ddot{z}_0}{\omega^2} \right) = -p_0 - \rho h_0 \ddot{z}_0 - \frac{\rho v \, w_m \, \omega}{h_0 \, \psi^2} P, \quad D^* = a_4 - a_7 \, b_1 - a_3 \, b_2. \tag{18}$$

In view of the boundary conditions Eq. (6), we present the solution of the Eqs. (17), (18) in the form of:

$$u = u_m \sum_{k=1}^{\infty} \frac{2k-1}{2\ell} \pi (R_k^0 + R_k(\tau)) \left[(-1)^{k-1} \xi - \sin \frac{2k-1}{2} \pi \xi \right], \tag{19}$$

$$\varphi = \varphi_m \sum_{k=1}^{\infty} \frac{2k-1}{2\ell} \pi (R_k^0 + R_k(\tau)) \left[(-1)^{k-1} \xi - \sin \frac{2k-1}{2} \pi \xi \right], \tag{20}$$

$$w = w_m W_0 = w_m \sum_{k=1}^{\infty} (R_k^0 + R_k(\tau)) \cos \frac{2k-1}{2} \pi \xi. \tag{21}$$

Here $R_k(\tau)$ is the harmonic time function, R_k^0 is the constant.

Bearing in mind Eqs. (13), (21) in the Eq. (19) and solving it we get:

$$w = \frac{p_0 \ell^4}{D^*} \sum_{k=1}^{\infty} 2(-1)^k \left(\frac{2}{(2k-1)\pi} \right)^5 \cos \frac{2k-1}{2\ell} \pi x + E_z \omega^2 \Pi_w(x, \omega) \sin(\omega t + \Psi_w(x, \omega)), \tag{22}$$

where the symbols are introduced:

$$\Pi_w(x, \omega) = \frac{m_0^*}{D^*} \left[\left(\sum_{k=1}^{\infty} \frac{4(-1)^k}{(2k-1)\pi} A_k \cos \left(\frac{2k-1}{2\ell} \pi x \right) \right)^2 \right.$$

$$\left. + \left(\sum_{k=1}^{\infty} \frac{4(-1)^k}{(2k-1)\pi} B_k \cos \left(\frac{2k-1}{2\ell} \pi x \right) \right)^2 \right]^{1/2},$$

$$\text{tg}(\Psi_w(x, \omega)) = \sum_{k=1}^{\infty} \frac{4(-1)^{k+1}}{(2k-1)\pi} A_k \cos \left(\frac{2k-1}{2\ell} \pi x \right) \Big/ \sum_{k=1}^{\infty} \frac{4(-1)^k}{(2k-1)\pi} B_k \cos \left(\frac{2k-1}{2\ell} \pi x \right),$$

$$A_k = \frac{a_{2k}}{(a_{1k}^2 + a_{2k}^2)}, \quad B_k = \frac{a_{1k}}{(a_{1k}^2 + a_{2k}^2)}, \quad a_{1k} = \left(\frac{2k-1}{2\ell} \pi \right)^4 - \frac{m_0 \omega^2}{D^*},$$

$$a_{2k} = \frac{2K_k \omega}{D^*}, \quad m_0^* = m_0 \left(1 + \frac{\rho h_0}{m_0} \right), \quad 2K_k = 12 \frac{\rho \nu}{h_0 \psi^2} \left[\frac{2}{(2k-1)\pi} \right]^2.$$

According to Eqs. (19), (20) and (17), we find the expressions for u and φ:

$$u = \frac{p_0 \ell^3 b_1}{D^*} \sum_{k=1}^{\infty} 2(-1)^k \left(\frac{2}{(2k-1)\pi} \right)^4 \left[(-1)^{k-1} \frac{x}{\ell} - \sin \frac{2k-1}{2\ell} \pi x \right]$$

$$+ E_z \omega^2 \Pi_u(x, \omega) \sin(\omega t + \Psi_u(x, \omega)), \tag{23}$$

$$\varphi = \frac{p_0 b_2 \ell^3}{D^*} \sum_{k=1}^{\infty} 2(-1)^k \left(\frac{2}{(2k-1)\pi} \right)^4 \left[(-1)^{k-1} \frac{x}{\ell} - \sin \frac{2k-1}{2\ell} \pi x \right]$$

$$+ E_z \omega^2 \Pi_\varphi(x, \omega) \sin(\omega t + \Psi_\varphi(x, \omega)), \tag{24}$$

where the symbols are introduced:

$$\Pi_u(x, \omega) = \frac{2b_1 m_0^*}{\ell D^*} \left[\left(\sum_{k=1}^{\infty} (-1)^k A_k \left[\frac{x}{\ell} (-1)^{k-1} - \sin \frac{2k-1}{2\ell} \pi x \right] \right)^2 \right.$$

$$\left. + \left(\sum_{k=1}^{\infty} (-1)^k B_k \left[\frac{x}{\ell} (-1)^{k-1} - \sin \frac{2k-1}{2\ell} \pi x \right] \right)^2 \right]^{1/2},$$

$$\Pi_\varphi(x, \omega) = \frac{b_2}{b_1} \Pi_u(x, \omega),$$

$$\operatorname{tg}\left(\Psi_{\varphi}(x,\omega)\right) = \operatorname{tg}\left(\Psi_u(x,\omega)\right) = \frac{\sum_{k=1}^{\infty}(-1)^{k+1} A_k \left(\frac{x}{\ell}(-1)^{k-1} - \sin\frac{2k-1}{2\ell}\pi x\right)}{\sum_{k=1}^{\infty}(-1)^{k} B_k \left(\frac{x}{\ell}(-1)^{k-1} - \sin\frac{2k-1}{2\ell}\pi x\right)}.$$

4. Summary and conclusion

The first component of Eqs. (22)-(24) corresponds to the loading from the constant pressure p_0, while the second one corresponds to the impact of the channel vibrating foundation. The functions $\Pi_w(x,\omega)$, $\Pi_u(x,\omega)$, $\Pi_{\varphi}(x,\omega)$ can be considered as the frequency dependent functions of deflection distribution, longitudinal displacement distribution and normal rotation angle distribution. The functions $\Psi_w(x,\omega)$, $\Psi_u(x,\omega)$, $\Psi_{\varphi}(x,\omega)$ can be considered as the frequency dependent functions of phase response distribution of values, mentioned above. In the case of the specified value of the coordinate x, these functions transform into amplitude-frequency characteristics and phase response characteristics in the assigned cross section of the three-layer beam. On the basis of these functions analyses it is possible to investigate the three-layer beam hydroelastic oscillations under foundation vibration. Thus, the obtained mathematical model can be used for investigating the three-layer beam resonance oscillations, where the beam is a wall of a channel with viscous incompressible liquid inside, as well as for studying its mode of deformation under vibration.

Acknowledgements

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