Hydroelastic oscillation of a plate resting on Pasternak foundation

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Abstract. The bending oscillations of the plate, resting in Pasternak foundation and interacting with a vibrating stamp through a thin layer of viscous incompressible liquid, are investigated. On the basis of hydroelasticity problem solution, the laws of the plate deflections and pressure in the liquid are found. The functions of the deflections amplitude distribution and liquid pressure along the channel are constructed. The obtained results allow to define oscillations resonance frequencies and to study viscous liquid interaction with elastic plates, resting on Pasternak foundation.

Keywords: hydroelastic oscillations, vibrating stamp, viscous liquid, plate, vibration, Pasternak foundation.

1. Introduction

The investigation of elastic construction elements oscillation with consideration as of their interaction with heterogeneous bodies, as of elastic pliability of the foundations, on which they are rested, presents an important issue of contemporary technology. For example, references [1-4] study the oscillations and stability of multi-layer beams and plates, rested on an elastic foundation under the influence of local and distributed loads of various origins. Winkler and Pasternak models are used for foundation reactions modeling. Reference [5] is devoted to numerical investigation of rectangular plates natural oscillations, the plates being fully plunged into motionless ideal liquid or floating on its free surface. The various variants of the plates fixing are considered. The analogous investigation for the case of rectangular plates interacting with the ideal liquid flow is made in reference [6]. The plate eigenfrequencies analysis is made and the flow critical velocities, corresponding to stability loss, are found. The issues of the chaotic oscillations of the plate, interacting with an ideal incompressible liquid flow, are studied in reference [7]. However, the liquid dumping properties, conditioned by its viscosity are excluded from consideration. Reference [8] deals with investigating vibrations of endlessly long beam on viscous liquid layer. The investigation of the cross oscillations of the elastic-fixed solid wall of the flat channel of finite sizes is made in [9]. Hydroelastic oscillations of cantilever beam, plunged into a viscous incompressible liquid, are investigated in reference [10]. The problem of vibrating discs dynamics interaction with a layer of viscous incompressible liquid between them is solved in reference [11]. The present paper studies the case when the discs are considered being solid, as well as the case with one of the discs being elastic. The analogous problem in a flat setting for the two vibrating plates of finite sizes is studied in reference [12]. Reference [13] investigates hydroelastic oscillation of the beam in a viscous liquid flow for the case of piezo-electric elements with the aim of obtaining energy from the flow. Reference [14] deals with solution of the problem of the bending hydroelastic oscillations of the plate, forming the narrow channel wall under the impact of the viscous liquid pulsating layer. The forced hydroelastic oscillations of the three-layered round plate interacting with viscous incompressible liquid layer under the channel foundation vibration are investigated in [15]. Alongside with this, the evaluation of foundation elasticity impact on hydroelastic oscillations is of theoretical and practical interest. Reference [16]

investigates the oscillations of the membrane, resting on Winkler elastic foundation, the membrane being on the bottom of reservoir, filled with an ideal incompressible liquid with a free surface. The hydroelastic oscillations of the rectangular plates, resting on Pasternak foundation and interacting with an ideal incompressible liquid with a free surface, are investigated in references [17-19]. The investigation of the oscillations of the plate, resting on the elastic Winkler foundation and interacting with viscous incompressible liquid pulsating layer, was made in references [20-22].

2. Statement of the problem

Let us consider the elastic plate 2 resting on Pasternak foundation and interacting with a vibrating stamp 1 through the viscous incompressible liquid layer 3 between them (Fig. 1). We will study the problem in a flat setting. Let us connect Cartesian coordinate Oxz with the center of the plate medium surface in an undisturbed state. The stamp movement takes place under the assigned harmonic law in a vertical plane. The plate possesses the thickness h_0, the length 2ℓ. It is simply supported on the edges. The viscous liquid fully fills a narrow channel, formed by a plate and a vibrating stamp, the medium thickness of a liquid layer in the channel being $\delta_0 \ll l$ and the plate deflection amplitudes being significantly less than δ_0. The liquid at the edges freely leaks out in the same liquid with a constant pressure level p_0. Taking into account the fact, that liquid viscosity consideration leads to quick going down of transition processes, we can exclude the initial processes impact from the very beginning [23]. Further, we will consider the stationary forced harmonic oscillations.
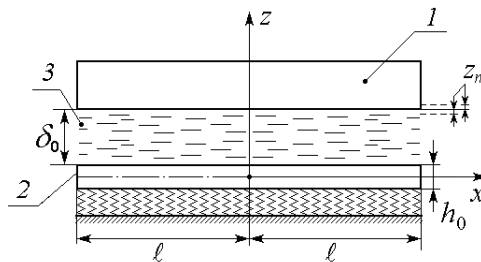

Fig. 1. A schematic diagram of a narrow slit channel on Winkler foundation

3. The theory

We will present the law of the stamp movement in the form of:

$$z = z_m f(\omega t), \ \ f(\omega t) = \sin\omega t, \tag{1}$$

where z_m is the stamp oscillation amplitude, ω is the oscillation frequency, t is the time.

Due to the channel narrowness, we will consider the liquid dynamics in it in terms of hydrodynamic lubrication theory, i.e. we investigate liquid creeping flow in the channel, which is described by the equation [24]:

$$\frac{1}{\rho}\frac{\partial p}{\partial x} = \nu\left(\frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_x}{\partial z^2}\right), \ \ \frac{1}{\rho}\frac{\partial p}{\partial z} = \nu\left(\frac{\partial^2 u_z}{\partial x^2} + \frac{\partial^2 u_z}{\partial z^2}\right), \ \ \frac{\partial u_x}{\partial x} + \frac{\partial u_z}{\partial z} = 0, \tag{2}$$

where u_x, u_z are liquid velocity projections on the coordinate axis, ρ is the density of the liquid, ν is the kinematic coefficient of the liquid viscosity, p is the pressure.

The boundary conditions (the conditions of liquid and walls shifts velocities coincidence, as well, as the conditions for the liquid pressure at edges) are corresponded to the Eq. (2):

$$u_x = \partial u/\partial t, \quad u_z = \partial w/\partial t, \quad at \ z = h_0/2 + w,$$
$$u_x = 0, \quad u_z = z_m \, df(\omega t)/dt, \quad at \ z = h_0/2 + \delta_0 + z_m f(\omega t), \tag{3}$$
$$p = p_0, \quad at \ x = \pm l, \tag{4}$$

where u, w are longitudinal movement and deflection plate laws.

The equation of the bending oscillations of the plate, resting on Pasternak foundation, can be written down, as in references [21, 25]:

$$D\frac{\partial^4 w}{\partial x^4} + \kappa w - \eta \frac{\partial^2 w}{\partial x^2} + \rho_0 h_0 \frac{\partial^2 w}{\partial t^2} = -q_{zz}|_{z=h_0/2+w}. \tag{5}$$

Here w is the plate deflection, D is the bending stiffness of plate, ρ_0 is the density of plate, κ is the foundation normal stiffness, η is the foundation shear coefficient, $q_{zz} = -p + 2\rho v \frac{\partial u_z}{\partial z}$ is the normal stress in a liquid layer [24].

Boundary conditions for Eq. (5) are the conditions of simply supported edges:

$$w = \partial^2 w/\partial x^2 = 0, \quad at \ x = \pm l. \tag{6}$$

Let us introduce dimensionless variables and small parameters into our consideration:

$$\xi = x/l, \quad \zeta = (z - 0{,}5h_0)/\delta_0, \quad \tau = \omega t, \quad u_z = w_m \omega U_\zeta, \quad u_x = (w_m \omega/\psi) U_\xi,$$
$$w = w_m W, \quad u = u_m U, \quad p = p_0 + P \, v\rho w_m \omega \psi^{-2}/\delta_0, \quad \lambda = w_m/\delta_0 \ll 1, \tag{7}$$
$$\psi = \delta_0/l \ll 1.$$

Taking into account Eq. (7), the hydrodynamic problem in dimensionless variables in zero approximation on ψ and λ will be written down, as:

$$\frac{\partial P}{\partial \xi} = \frac{\partial^2 U_\xi}{\partial \zeta^2}, \quad \frac{\partial P}{\partial \zeta} = 0, \quad \frac{\partial U_\xi}{\partial \xi} + \frac{\partial U_\zeta}{\partial \zeta} = 0. \tag{8}$$

With boundary conditions:

$$U_\xi = 0, \quad U_\zeta = (z_m/w_m) \, df(\tau)/d\tau, \quad at \ \zeta = 1, \quad U_\xi = \psi(u_m/w_m) \, \partial U/\partial \tau,$$
$$U_\zeta = \partial W/\partial \tau, \quad at \ \zeta = 0, \quad P = 0, \quad at \ \xi = \pm 1. \tag{9}$$

The normal stress in the liquid layer in the variables Eq. (7) is written down, as:

$$q_{zz} = -p_0 - \frac{v\rho w_m \omega}{\psi^2 \delta_0} P. \tag{10}$$

Further, in the course of solving the problem of Eqs. (8)-(9), we will assume, that $u_m/w_m = O(1)$, i.e. we consider, that in boundary conditions Eq. (9) will be $U_\xi = \psi(u_m/w_m) \, \partial U/\partial \tau = o(1)$. As a result, we get:

$$U_\xi = \frac{\zeta^2 - \zeta}{2} \frac{\partial P}{\partial \xi}, \quad U_\zeta = \frac{\partial W}{\partial \tau} + (2\zeta^3 - 3\zeta^2)\left(\frac{\partial W}{\partial \tau} - \frac{z_m}{w_m}\frac{df}{d\tau}\right). \tag{11}$$

As well, as the equations for the pressure:

$$\frac{\partial^2 P}{\partial \xi^2} = -12\left(\frac{\partial W}{\partial \tau} - \frac{z_m}{w_m}\frac{df}{d\tau}\right). \tag{12}$$

By integrating Eq. (12) and satisfying the boundary conditions for the pressure Eq. (9), we find:

$$P = 6(\xi^2 - 1)\frac{z_m}{w_m}\frac{df}{d\tau} - 12\int_{-1}^{\xi}\int \partial W/\partial\tau\, d\xi d\xi + 6(\xi + 1)\int_{-1}^{1}\int \partial W/\partial\tau\, d\xi d\xi. \tag{13}$$

Taking into account the boundary conditions Eq. (6), we present the form of the plate elastic deflections as the series of Sturm-Liouville problem eigenfunctions:

$$w = w_m W = w_m \sum_{k=1}^{\infty}\left(R_k^0 + R_k(\tau)\right)\cos((2k-1)\pi\xi/2). \tag{14}$$

Here $R_k(\tau)$ is the harmonic time function, R_k^0 is the constant.

As a result, dimensionless pressure Eq. (13) with consideration of the deflection chosen form Eq. (14), takes form of:

$$P = 12\sum_{k=1}^{\infty}\left[\frac{z_m}{w_m}\frac{df}{d\tau}\frac{4(-1)^k}{(2k-1)\pi}\left(\frac{2}{(2k-1)\pi}\right)^2 + \left(\frac{2}{(2k-1)\pi}\right)^2\frac{dR_k}{d\tau}\right]\cos\frac{2k-1}{2}\pi\xi. \tag{15}$$

By substituting Eq. (15) in the expression for the normal stress Eq. (10) and disintegrating the constant pressure p_0 in the series of Sturm-Liouville problem eigenfunctions, we get:

$$\begin{aligned}
q_{zz} &= p_0\sum_{k=1}^{\infty}\frac{4(-1)^k}{(2k-1)\pi}\cos\frac{2k-1}{2}\pi\xi \\
&- \frac{\rho v w_m \omega}{\delta_0\psi^2}\left[12\sum_{k=1}^{\infty}\left(\frac{2}{(2k-1)\pi}\right)^2\frac{dR_k}{d\tau}\cos\frac{2k-1}{2}\pi\xi\right. \\
&\left. + 12\frac{z_m}{w_m}\frac{df}{d\tau}\sum_{k=1}^{\infty}\frac{4(-1)^k}{(2k-1)\pi}\left(\frac{2}{(2k-1)\pi}\right)^2\cos\frac{2k-1}{2}\pi\xi\right].
\end{aligned} \tag{16}$$

Taking into account Eqs. (14), (16) we write down the plate dynamics equations in the form of:

$$\begin{aligned}
&\left(\frac{D}{l^4}\left(\frac{(2k-1)\pi}{2}\right)^4 + \kappa + \frac{\eta}{l^2}\left(\frac{(2k-1)\pi}{2}\right)^2\right)w_m(R_k^0 + R_k) + \rho_0 h_0\omega^2 w_m\frac{d^2R_k}{d\tau^2} \\
&+ 12\frac{\rho v\omega}{\delta_0\psi^2}\left(\frac{2}{(2k-1)\pi}\right)^2 w_m\frac{dR_k}{d\tau} \\
&= p_0\frac{4(-1)^k}{(2k-1)\pi} - 12\frac{\rho v\omega}{\delta_0\psi^2}\frac{4(-1)^k}{(2k-1)\pi}\left(\frac{2}{(2k-1)\pi}\right)^2 z_m\frac{df}{d\tau}.
\end{aligned} \tag{17}$$

Due to Eq. (17) linearity for the constant pressure component, we get the following expression:

$$w_m R_k^0 = p_0\frac{4(-1)^k}{(2k-1)\pi}\left(\frac{D}{l^4}\left(\frac{(2k-1)\pi}{2}\right)^4 + \frac{\eta}{l^2}\left(\frac{(2k-1)\pi}{2}\right)^2 + \kappa\right)^{-1}. \tag{18}$$

The solution of Eq. (17) for the stationary harmonic oscillations regime has the form of:

$$w_m R_k = -z_m \left[\frac{K_k^z K_k^w \omega^2}{(\widetilde{D}_k - \rho_0 h_0 \omega^2)^2 + (K_k^w \omega)^2} f(\tau) + \frac{K_k^z \omega (\widetilde{D}_k - \rho_0 h_0 \omega^2)}{(\widetilde{D}_k - \rho_0 h_0 \omega^2)^2 + (K_k^w \omega)^2} \frac{df(\tau)}{d\tau} \right],$$

$$\widetilde{D}_k = \frac{D}{l^4} \left(\frac{(2k-1)\pi}{2} \right)^4 + \frac{\eta}{l^2} \left(\frac{(2k-1)\pi}{2} \right)^2 + \kappa, \quad K_k^z = 12 \frac{\rho v}{\delta_0 \psi^2} \frac{4(-1)^k}{(2k-1)\pi} \left(\frac{2}{(2k-1)\pi} \right)^2,$$

$$K_k^w = 12 \frac{\rho v}{\delta_0 \psi^2} \left(\frac{2}{(2k-1)\pi} \right)^2. \tag{19}$$

Finally, the law of the plate hydroelastic oscillations, caused by the stamp movement and with consideration of Eq. (18) and (19), takes the form:

$$w = \frac{p_0 l^4}{D} \sum_{k=1}^{\infty} \frac{4(-1)^k}{(2k-1)\pi} \left(\left(\frac{(2k-1)\pi}{2} \right)^4 + \frac{\eta l^2}{D} \left(\frac{(2k-1)\pi}{2} \right)^2 + \frac{\kappa l^4}{D} \right)^{-1}$$

$$\cdot \cos \left(\frac{2k-1}{2l} \pi x \right) - z_m A(x, \omega) \sin(\omega t + \varphi(x, \omega)),$$

$$A(x, \omega) = \sqrt{C^2 + B^2}, \quad \varphi(x, \omega) = \text{arctg}(B/C), \tag{20}$$

$$C = \sum_{k=1}^{\infty} \frac{K_k^z K_k^w \omega^2}{(\widetilde{D}_k - \rho_0 h_0 \omega^2)^2 + (K_k^w \omega)^2} \cos \frac{2k-1}{2l} \pi x,$$

$$B = \sum_{k=1}^{\infty} \frac{K_k^z \omega (\widetilde{D}_k - \rho_0 h_0 \omega^2)}{(\widetilde{D}_k - \rho_0 h_0 \omega^2)^2 + (K_k^w \omega)^2} \cos \frac{2k-1}{2l} \pi x.$$

By substituting Eq. (19) in Eq. (15) and taking into account Eq. (7), the pressure in a liquid layer between the stamp and the plate can be written down as:

$$p = p_0 + z_m \Pi(x, \omega) \sin \left(\omega t + \varphi_p(x, \omega) \right), \quad \Pi(x, \omega) = \sqrt{S^2 + Q^2},$$

$$\varphi_p(x, \omega) = \text{arctg}(Q/S), \quad Q = \sum_{k=1}^{\infty} K_k^z \omega \left(1 - \frac{(K_k^w \omega)^2}{(\widetilde{D}_k - \rho_0 h_0 \omega^2)^2 + (K_k^w \omega)^2} \right) \cos \frac{2k-1}{2l} \pi x, \tag{21}$$

$$S = \sum_{k=1}^{\infty} \frac{K_k^w K_k^z \omega^2 (\widetilde{D}_k - \rho_0 h_0 \omega^2)}{(\widetilde{D}_k - \rho_0 h_0 \omega^2)^2 + (K_k^w \omega)^2} \cos \frac{2k-1}{2l} \pi x.$$

4. Summary and conclusion

The obtained analytical solution of the problems allows making the following conclusions. The first component of the expression for the deflection Eq. (20) is conditioned by static pressure in liquid p_0, the second component being a dynamic deflection, which is conditioned by the plate-stamp interaction through viscous liquid layer. The value of second deflection component is defined by the function $A(x, \omega)$, which may be considered as a frequency dependent amplitude sag distribution along the channel. The analogous remarks can be made in relation to the law of pressure change in the channel Eq. (21). The value of the second pressure component is defined by the function $\Pi(x, \omega)$, which is a frequency dependent amplitude distribution of dynamic pressure along the channel. The functions $\varphi(x, \omega)$, $\varphi_p(x, \omega)$ are frequency dependent functions of the plate deflection phase shift distribution and the pressure phase shift distribution along the channel, correspondingly.

The investigation of the functions behavior depending on oscillations frequency allows studying dynamic processes in the considered oscillation system. The analysis of the obtained expression for the plate deflection Eq. (20) makes it possible to assume that the coefficients, defining Pasternak foundation reaction, will influence the plate static deflection, as well as

deflection dynamic amplitude. Also, the transition from Pasternak foundation to Winkler one is possible in terms of the obtained solution, the coefficient of the foundation shift η being zero. Thus, the obtained results can be used for modeling and analyzing hydroelastic oscillations of elastic elements of the constructions, resting on the foundations and interacting with viscous liquid.

Acknowledgement

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