35. Dynamic problem in 3D thermoelastic half-space with rotation in context of G-N type II and type III

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Abstract. In this paper, comparison between G-N model of type II (without energy dissipation) and G-N model of type III (with energy dissipation) has been shown in a three dimensional thermoelastic half space with rotation subjected to time dependent heat source on the traction free boundary. Eigenvalue methodology has been adopted to solve the equations resulting from the application of the Normal mode analysis to the non-dimensional coupled equations. Variation of the numerically computed values of thermal stresses and temperature with and without rotation has been illustrated graphically.

Keywords: anaisotropic half space, G-N model II and III, normal mode analysis, eigenvalue approach.

Nomenclature

c_{ij}	Material constant
β_{ij}	Thermal elastic coupling tensor
δ_{ij}	Kronecker delta
u_i	Displacement components
e_{ij}	Strain tensor
τ_{ij}	Stress components
ρ	Mass density

1. Introduction

The inconsistency of heat conduction equation of classical uncoupled theory of thermoelasticity with the experimental results is due to the fact that i. no elastic term is included to account for elastic changes producing heat effects; ii. parabolic nature of heat conduction equation indicating infinite speed of propagation of heat waves means that thermal disturbances (with infinite speed) and elastic disturbances (with finite speed) from the classical theory of thermoelasticity, are coupled together. This suggests that every solution of the equations has a part which extends to infinity.

Biot [1] developed a theory of irreversible thermodynamics and gave a satisfactory derivation of the linear theory of coupled thermoelasticity. In order to obtain a wave type heat conduction equation the concept of generalized thermoelasticity was introduced modifying CCTE and later extended by Dhaliwal and Sherief [3] for anisotropic body, and the uniqueness of the solutions was proved by Ignaczak [2, 4]. Green-Naghdi proposed a generalized thermoelasticity by modifying the energy equation. There are three types of constitutive relations in G-N model [6-8]. Type-I leads to classical heat conduction equation. Type-II provides solutions for thermal waves propagating finite speed without energy dissipation (TEWOED) and type-III also confirms propagation of thermal waves of finite speed with energy dissipation (TEWED). Several investigations with these extensions have been studied by Abd-Alla and Abo-Dahab [11], Kar and Kanoria [9] and Yousef [10]. Pal et al. [5, 14] studied the effect of homogeneity of the surface waves in anisotropic media.

In this paper, the stress distributions and temperature variation has been depicted in an anisotropic triclinic half space for G-N model II and III both with rotation.

2. Basic equations

For a linear thermoelastic anisotropic body subjected to rotation the field equations are as follows:

The equation of motion in the absence of inner heat source:

$$\tau_{ij,i} = \rho[\ddot{u}_i + \bar{\Omega}\times(\bar{\Omega}\times\bar{u}_i) + (2\bar{\Omega}\times\dot{\bar{u}}_i)]. \tag{1}$$

Heat – conduction equation in absence of body force:

$$k^*\theta_{,ij} + k_{ij}\dot{\theta}_{,ij} = \rho c_e\ddot{\theta} + \theta_0\beta_{ij}\ddot{u}_{i,j}. \tag{2}$$

The Duhamel-Neumann stress-strain relations are:

$$\tau_{ij} = c_{ijkl}e_{kl} - \beta_{ij}\theta\delta_{ij}\,, \tag{3}$$

where $\beta_{ij} = c_{ijkl}\alpha_{kl}$ $(i, j, k, l = 1, 2, 3)$.

Strain – displacement relation:

$$e_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right),\ \ u_{i,j} = \frac{\partial u_i}{\partial x_j}, \tag{4}$$

where the differentiation with respect to space variable x_i is denoted by and that with respect to time is denoted by notation.

3. Formulation of the problem

Let us consider a linear anisotropic thermoelastic half space within $\{(x_1, x_2, x_3)\colon 0 \le x_1 < \infty,$ $0 \le x_2 < \infty, 0 \le x_3 < \infty\}$ for a time dependent heat source on the boundary plane to the surface $x_1 = 0$. The surface $x_1 = 0$ is assumed to be traction free and the body is assumed initially at rest. The components of displacement vectors of three dimensional plane waves in anisotropic elastic medium, are given as:

$$u_i = u_i(x_1, x_2, x_3, t)\,,\quad i = 1, 2, 3, \tag{5}$$

where t is the time variable and x_i, $(i = 1, 2, 3)$ denotes the respective orthogonal Cartesian co-ordinate axes. The elastic medium is now considered as rotating uniformly with an angular velocity $\Omega = \Omega\hat{n}$, where $\hat{n}$ is the unit vector along the direction of rotation. The equation of motion of the rotating frame contains two additional terms: $(\Omega\times(\Omega\times u)$ representing the centripetal acceleration due to time varying motion only and $(2\Omega\times\dot{u})$ representing the coriolis acceleration and $\hat{n} = (1, 0, 0)$.

Using Hook's law, the stress- strain- temperature relations can be written as follows:

$$\tau_{11} = c_{11}e_{11} + c_{12}e_{22} + c_{13}e_{33} + 2(c_{14}e_{23} + c_{15}e_{13} + c_{16}e_{12}) - \beta_{11}\theta, \tag{6a}$$
$$\tau_{22} = c_{21}e_{11} + c_{22}e_{22} + c_{23}e_{33} + 2(c_{24}e_{23} + c_{25}e_{13} + c_{26}e_{12}) - \beta_{22}\theta, \tag{6b}$$
$$\tau_{33} = c_{31}e_{11} + c_{32}e_{22} + c_{33}e_{33} + 2(c_{34}e_{23} + c_{35}e_{13} + c_{36}e_{12}) - \beta_{33}\theta, \tag{6c}$$
$$\tau_{23} = c_{41}e_{11} + c_{42}e_{22} + c_{43}e_{33} + 2(c_{44}e_{23} + c_{45}e_{13} + c_{46}e_{12}), \tag{6d}$$
$$\tau_{13} = c_{51}e_{11} + c_{52}e_{22} + c_{53}e_{33} + 2(c_{54}e_{23} + c_{55}e_{13} + c_{56}e_{12}), \tag{6e}$$
$$\tau_{12} = c_{61}e_{11} + c_{62}e_{22} + c_{63}e_{33} + 2(c_{64}e_{23} + c_{65}e_{13} + c_{66}e_{12}). \tag{6f}$$

In absence of inner heat source and body force the equation of motion are given as:

$$\tau_{11,1}+\tau_{12,2}+\tau_{13,3}=\rho\ddot{u}_1, \tag{7a}$$
$$\tau_{21,1}+\tau_{22,2}+\tau_{23,3}=\rho\ddot{u}_2-\Omega^2u_2-2\Omega\dot{u}_3, \tag{7b}$$
$$\tau_{31,1}+\tau_{32,2}+\tau_{33,3}=\rho\ddot{u}_3-\Omega^2u_3+2\Omega\dot{u}_2. \tag{7c}$$

With the help of Eqs. (5) and (6), the equations of motions Eq. (7) become:

$$\begin{aligned}&\left(c_{11}u_{1,11}+c_{66}u_{1,22}+c_{55}u_{1,33}\right)+2\left(c_{16}u_{1,12}+c_{15}u_{1,13}+c_{56}u_{1,23}\right)\\&\quad+\left(c_{16}u_{2,11}+c_{26}u_{2,22}+c_{45}u_{2,33}+(c_{12}+c_{66})u_{2,12}+(c_{14}+c_{56})u_{2,13}\right.\\&\quad\left.+(c_{46}+c_{25})u_{2,23}\right)\\&\quad+\left(c_{15}u_{3,11}+c_{46}u_{3,22}+c_{35}u_{3,33}+(c_{14}+c_{56})u_{3,12}+(c_{13}+c_{55})u_{3,13}\right.\\&\quad\left.+(c_{36}+c_{45})u_{3,23}\right)-\beta_{11}\theta_{,1}=\rho\ddot{u}_1,\end{aligned} \tag{8a}$$

$$\begin{aligned}&\left(c_{16}u_{1,11}+c_{26}u_{1,22}+c_{45}u_{1,33}+(c_{12}+c_{66})u_{1,12}+(c_{14}+c_{56})u_{1,13}+(c_{46}+c_{25})u_{1,23}\right)\\&\quad+\left(c_{66}u_{2,11}+c_{22}u_{2,22}+c_{44}u_{2,33}\right)+2\left(c_{26}u_{2,12}+c_{46}u_{2,13}+c_{24}u_{2,23}\right)\\&\quad+\left(c_{56}u_{3,11}+c_{24}u_{3,22}+c_{34}u_{3,33}+(c_{46}+c_{25})u_{3,12}+(c_{36}+c_{45})u_{3,13}\right.\\&\quad\left.+(c_{23}+c_{44})u_{3,23}\right)-\beta_{22}\theta_{,2}=\rho\ddot{u}_2-\Omega^2u_2-2\Omega\dot{u}_3,\end{aligned} \tag{8b}$$

$$\begin{aligned}&\left(c_{15}u_{1,11}+c_{46}u_{1,22}+c_{35}u_{1,33}+(c_{14}+c_{56})u_{1,12}+(c_{13}+c_{55})u_{1,13}+(c_{36}+c_{45})u_{1,23}\right)\\&\quad+\left(c_{56}u_{2,11}+c_{24}u_{2,22}+c_{34}u_{2,33}+(c_{46}+c_{25})u_{2,12}+(c_{36}+c_{45})u_{2,13}\right.\\&\quad\left.+(c_{23}+c_{44})u_{2,23}\right)+\left(c_{55}u_{3,11}+c_{44}u_{3,22}+c_{33}u_{3,33}\right)\\&\quad+2\left(c_{45}u_{3,12}+c_{35}u_{3,13}+c_{34}u_{3,23}\right)-\beta_{33}\theta_{,3}=\rho\ddot{u}_3-\Omega^2u_3+2\Omega\dot{u}_2.\end{aligned} \tag{8c}$$

The generalized heat-conduction Eq. (2) is written as:

$$\begin{aligned}&\left(k^*+k_{11}\frac{\partial}{\partial t}\right)\theta_{,11}+\left(k^*+k_{22}\frac{\partial}{\partial t}\right)\theta_{,22}+\left(k^*+k_{33}\frac{\partial}{\partial t}\right)\theta_{,33}\\&\quad=\rho c_e\ddot{\theta}+\theta_0\left(\beta_{11}\ddot{u}_{1,1}+\beta_{22}\ddot{u}_{2,2}+\beta_{33}\ddot{u}_{3,3}\right).\end{aligned} \tag{9}$$

The following non- dimensional variables are introduced to transform the above equations in non-dimensional form:

$$x'_i=\frac{1}{l}x_i,\ \ u'_i=\frac{c_{11}}{l\beta_{11}\theta_0}u_i,\ \ t'=\sqrt{\frac{c_{11}}{\rho}}\frac{1}{l}t,\ \ \theta'=\frac{\theta}{\theta_0},\ \ c_1^2=\frac{c_{11}}{\rho},\ \ \Omega=\frac{c_1}{l}\Omega', \tag{10}$$

where l is some standard length.

Using Eq. (10), the non dimensional forms of the Eqs. (8a)-(9) reduces to (omitting primes for convenience):

$$\begin{aligned}&\left(u_{1,11}+\frac{c_{66}}{c_{11}}u_{1,22}+\frac{c_{55}}{c_{11}}u_{1,33}\right)+2\left(\frac{c_{16}}{c_{11}}u_{1,12}+\frac{c_{15}}{c_{11}}u_{1,13}+\frac{c_{56}}{c_{11}}u_{1,23}\right)\\&\quad+\left(\frac{c_{16}}{c_{11}}u_{2,11}+\frac{c_{26}}{c_{11}}u_{2,22}+\frac{c_{45}}{c_{11}}u_{2,33}+\left(\frac{c_{12}+c_{66}}{c_{11}}\right)u_{2,12}+\left(\frac{c_{14}+c_{56}}{c_{11}}\right)u_{2,13}\right.\\&\quad\left.+\left(\frac{c_{46}+c_{25}}{c_{11}}\right)u_{2,23}\right)\\&\quad+\left(\frac{c_{15}}{c_{11}}u_{3,11}+\frac{c_{46}}{c_{11}}u_{3,22}+\frac{c_{35}}{c_{11}}u_{3,33}+\left(\frac{c_{14}+c_{56}}{c_{11}}\right)u_{3,12}+\left(\frac{c_{13}+c_{55}}{c_{11}}\right)u_{3,13}\right.\\&\quad\left.+\left(\frac{c_{36}+c_{45}}{c_{11}}\right)u_{3,23}\right)-\theta_{,1}=\ddot{u}_1,\end{aligned} \tag{11a}$$

$$\begin{aligned}
&\left(\frac{c_{16}}{c_{11}}u_{1,11}+\frac{c_{26}}{c_{11}}u_{1,22}+\frac{c_{45}}{c_{11}}u_{1,33}+\left(\frac{c_{12}+c_{66}}{c_{11}}\right)u_{1,12}+\left(\frac{c_{14}+c_{56}}{c_{11}}\right)u_{1,13}\right.\\
&\quad\left.+\left(\frac{c_{46}+c_{25}}{c_{11}}\right)u_{1,23}\right)+\left(\frac{c_{66}}{c_{11}}u_{2,11}+\frac{c_{22}}{c_{11}}u_{2,22}+\frac{c_{44}}{c_{11}}u_{2,33}\right)\\
&\quad+2\left(\frac{c_{26}}{c_{11}}u_{2,12}+\frac{c_{46}}{c_{11}}u_{2,13}+\frac{c_{24}}{c_{11}}u_{2,23}\right)\\
&\quad+\left(\frac{c_{56}}{c_{11}}u_{3,11}+\frac{c_{24}}{c_{11}}u_{3,22}+\frac{c_{34}}{c_{11}}u_{3,33}+\left(\frac{c_{46}+c_{25}}{c_{11}}\right)u_{3,12}+\left(\frac{c_{36}+c_{45}}{c_{11}}\right)u_{3,13}\right.\\
&\quad\left.+\left(\frac{c_{23}+c_{44}}{c_{11}}\right)u_{3,23}\right)-\beta_2\theta_{,2}=\ddot{u}_2-\Omega^2u_2-2\Omega\dot{u}_3,
\end{aligned} \tag{11b}$$

$$\begin{aligned}
&\left(\frac{c_{15}}{c_{11}}u_{1,11}+\frac{c_{46}}{c_{11}}u_{1,22}+\frac{c_{35}}{c_{11}}u_{1,33}+\left(\frac{c_{14}+c_{56}}{c_{11}}\right)u_{1,12}+\left(\frac{c_{13}+c_{55}}{c_{11}}\right)u_{1,13}\right.\\
&\quad\left.+\left(\frac{c_{36}+c_{45}}{c_{11}}\right)u_{1,23}\right)\\
&\quad+\left(\frac{c_{56}}{c_{11}}u_{2,11}+\frac{c_{24}}{c_{11}}u_{2,22}+\frac{c_{34}}{c_{11}}u_{2,33}+\left(\frac{c_{46}+c_{25}}{c_{11}}\right)u_{2,12}+\left(\frac{c_{36}+c_{45}}{c_{11}}\right)u_{2,13}\right.\\
&\quad\left.+\left(\frac{c_{23}+c_{44}}{c_{11}}\right)u_{2,23}\right)+\left(\frac{c_{55}}{c_{11}}u_{3,11}+\frac{c_{44}}{c_{11}}u_{3,22}+\frac{c_{33}}{c_{11}}u_{3,33}\right)\\
&\quad+2\left(\frac{c_{45}}{c_{11}}u_{3,12}+\frac{c_{35}}{c_{11}}u_{3,13}+\frac{c_{34}}{c_{11}}u_{3,23}\right)-\beta_3\theta_{,3}=\ddot{u}_3-\Omega^2u_3+2\Omega\dot{u}_2,
\end{aligned} \tag{11c}$$

$$\begin{aligned}
&\left(\epsilon_3+\epsilon_4\frac{\partial}{\partial t}\right)\theta_{,11}+\left(\epsilon_3+\epsilon_4k_2\frac{\partial}{\partial t}\right)\theta_{,22}+\left(\epsilon_3+\epsilon_4k_3\frac{\partial}{\partial t}\right)\theta_{,33}\\
&\quad=\ddot{\theta}+\left(\epsilon_0\ddot{u}_{1,1}+\epsilon_1\ddot{u}_{2,2}+\epsilon_2\ddot{u}_{3,3}\right),
\end{aligned} \tag{12}$$

where:

$$[\epsilon_0,\epsilon_1,\epsilon_2]=\frac{\beta_{11}\theta_0}{\rho c_e c_{11}}[\beta_{11},\beta_{22},\beta_{33}],\quad [\epsilon_3,\epsilon_4]=\frac{1}{\rho c_e}\left[\frac{k^*}{c_1^2},\frac{k_{11}c_1}{l}\right],$$
$$k_2=\frac{k_{22}}{k_{11}},\quad k_3=\frac{k_{33}}{k_{11}},\quad \beta_2=\frac{\beta_{22}}{\beta_{11}},\quad \beta_3=\frac{\beta_{33}}{\beta_{11}}.$$

Non – dimensional stress components can be calculated as:

$$\tau_{11}=\frac{1}{c_{11}}[c_{11}e_{11}+c_{12}e_{22}+c_{13}e_{33}+2(c_{14}e_{23}+c_{15}e_{13}+c_{16}e_{12})]-\theta, \tag{13a}$$

$$\tau_{22}=\frac{1}{c_{11}}[c_{21}e_{11}+c_{22}e_{22}+c_{23}e_{33}+2(c_{24}e_{23}+c_{25}e_{13}+c_{26}e_{12})]-\beta_2\theta, \tag{13b}$$

$$\tau_{33}=\frac{1}{c_{11}}[c_{31}e_{11}+c_{32}e_{22}+c_{33}e_{33}+2(c_{34}e_{23}+c_{35}e_{13}+c_{36}e_{12})]-\beta_3\theta, \tag{13c}$$

$$\tau_{23}=\frac{1}{c_{11}}[c_{41}e_{11}+c_{42}e_{22}+c_{43}e_{33}+2(c_{44}e_{23}+c_{45}e_{13}+c_{46}e_{12})], \tag{13d}$$

$$\tau_{13}=\frac{1}{c_{11}}[c_{51}e_{11}+c_{52}e_{22}+c_{53}e_{33}+2(c_{54}e_{23}+c_{55}e_{13}+c_{56}e_{12})], \tag{13e}$$

$$\tau_{12}=\frac{1}{c_{11}}[c_{61}e_{11}+c_{62}e_{22}+c_{63}e_{33}+2(c_{64}e_{23}+c_{65}e_{13}+c_{66}e_{12})]. \tag{13f}$$

4. Normal mode analysis

The physical variables are decomposed in terms of normal modes to obtain the solution of (11) and (12) (Sarkar and Lahiri [13]) in the following form:

$$\left(u_i,\theta,\tau_{ij}\right)[x_1,x_2,x_3,t]=\left(u_i^*,\theta^*,\tau_{ij}^*\right)(x_i)e^{\omega t+i(ax_2+bx_3)}, \tag{14}$$

where $i=\sqrt{-1}$, ω being angular frequency and a, b are the wave numbers are a and b along x_2 and x_3 directions respectively.

Using Eq. (14) in Eqs. (11)-(13), we obtained (omitting '*' for convenience):

$$\begin{aligned}&u_{1,11}+a_{11}u_{1,1}+a_{12}u_1+a_{21}u_{2,11}+a_{22}u_{2,1}+a_{23}u_2\\&\quad+a_{31}u_{3,11}+a_{32}u_{3,1}+a_{33}u_3-\theta_{,1}=0,\end{aligned} \tag{15}$$

$$\begin{aligned}&b_{11}u_{1,11}+b_{12}u_{1,1}+b_{13}u_1+u_{2,11}+b_{21}u_{2,1}+b_{22}u_2\\&\quad+b_{31}u_{3,11}+b_{32}u_{3,1}+b_{33}u_3-b_{34}\theta_{,1}=0,\end{aligned} \tag{16}$$

$$\begin{aligned}&m_{11}u_{1,11}+m_{12}u_{1,1}+m_{13}u_1+m_{21}u_{2,11}+m_{22}u_{2,1}\\&\quad+m_{23}u_2+u_{3,11}+m_{31}u_{3,1}+m_{32}u_3-m_{33}\theta_{,1}=0,\end{aligned} \tag{17}$$

$$d_{41}u_{1,1}+d_{44}u_2+d_{46}u_3+d_{48}\theta=\theta_{,11}, \tag{18}$$

$$\tau_{11}=u_{1,1}+h_{12}u_{2,1}+h_{13}u_{3,1}+h_{14}u_1+h_{15}u_2+h_{16}h_3-\theta, \tag{19}$$

$$\tau_{22}=h_{21}u_{1,1}+h_{22}u_{2,1}+h_{23}u_{3,1}+h_{24}u_1+h_{25}u_2+h_{26}u_3-\beta_2\theta, \tag{20}$$

$$\tau_{33}=h_{31}u_{1,1}+h_{32}u_{2,1}+h_{33}u_{3,1}+h_{34}u_1+h_{35}u_2+h_{36}u_3-\beta_3\theta, \tag{21}$$

$$\tau_{23}=h_{41}u_{1,1}+h_{42}u_{2,1}+h_{43}u_{3,1}+h_{44}u_1+h_{45}u_2+h_{46}u_3, \tag{22}$$

$$\tau_{13}=h_{51}u_{1,1}+h_{52}u_{2,1}+h_{53}u_{3,1}+h_{54}u_1+h_{55}u_2+h_{56}u_3, \tag{23}$$

$$\tau_{12}=h_{61}u_{1,1}+h_{62}u_{2,1}+h_{63}u_{3,1}+h_{64}u_1+h_{65}u_2+h_{66}u_3, \tag{24}$$

where a_{ij}, b_{ij}, m_{ij}, d_{ij} and h_{ij} $(i,j=1,2,3)$ are given in Appendix I.

Equations (15)- (18) can be written in the vector- matrix differential equations (Sarkar and Lahiri [13]) as follows:

$$\frac{d\tilde{A}}{dx}=\tilde{A}\,\tilde{V}, \tag{25}$$

where:

$$\tilde{V}=\left[u_1,u_2,u_3,\theta,u_{1,1},u_{2,1},u_{3,1},\theta_{,1}\right],$$
$$\tilde{A}=\begin{bmatrix}L_{11} & L_{12}\\ L_{21} & L_{22}\end{bmatrix},$$

where L_{11} and L_{12} are null matrix and identity matrix of order 4×4 and L_{21} and L_{22} are given in the Appendix I.

5. Solution of the vector- matrix differential equation: eigenvalue approach

Applying eigenvalue approach method as in Santra et al. [15] to solve the vector-matrix differential equation (25), we get the characteristic equation of the matrix $\tilde{A}$ as:

$$\left|\tilde{A}-\lambda I\right|=0. \tag{26}$$

The roots of the Eq. (26) are of the form:

$$\lambda=\pm\lambda_i,\quad (i=1,2,3,4).$$

The eigenvector $\tilde{X}$ corresponding to the eigenvalue λ can be calculated as:

$$\tilde{X} = \begin{bmatrix} \delta_1 = (f_{24}f_{13} - f_{14}f_{23})(f_{22}f_{33} - f_{32}f_{23}) - (f_{34}f_{23} - f_{24}f_{33})(f_{12}f_{23} - f_{22}f_{13}) \\ \delta_2 = (f_{34}f_{23} - f_{24}f_{33})(f_{11}f_{23} - f_{21}f_{13}) - (f_{24}f_{13} - f_{14}f_{23})(f_{21}f_{33} - f_{31}f_{23}) \\ \delta_3 = (f_{12}f_{21} - f_{11}f_{22})(f_{21}f_{34} - f_{31}f_{24}) - (f_{22}f_{31} - f_{21}f_{32})(f_{11}f_{24} - f_{14}f_{21}) \\ \delta_4 = (f_{11}f_{23} - f_{21}f_{13})(f_{22}f_{33} - f_{32}f_{23}) - (f_{12}f_{23} - f_{22}f_{13})(f_{21}f_{33} - f_{31}f_{23}) \\ \lambda\delta_1 \\ \lambda\delta_2 \\ \lambda\delta_3 \\ \lambda\delta_4 \end{bmatrix}, \tag{27}$$

where f_{ij}, $(i, j = 1, 2, 3)$ are given in the Appendix I.

The eigenvector $\tilde{X}_i$ $[i = 1(1)8]$ corresponding to the eigenvalue $\lambda = \lambda_i$ $[i = 1(1)8]$ can be calculated from equation (27). For our further reference, we use the following notations:

$$\tilde{X}_i = \begin{cases} [\tilde{X}]_{\lambda=\lambda_{\frac{i+1}{2}}}, & i = 1(2)7, \\ [\tilde{X}]_{\lambda=\lambda_{-\frac{i}{2}}}, & i = 2(2)8. \end{cases} \tag{28}$$

As in Lahiri et al. [12], the general solution of Eq. (25) which is regular as $x_i \to +\infty$ can be written as:

$$\tilde{V} = \sum_{i=1}^{4} A_i \tilde{X}_{2i} e^{-\lambda_i x_1}, \quad x_1 \geq 0, \tag{29}$$

where the arbitrary constants A_i are to be determined from the boundary conditions of the problem and because of regularity condition of the solution at $+\infty$ the terms containing exponential of growing nature in the space variables x_i have been neglected.

Thus, the field variables can be written from the Eq. (29) for $x_1 \geq 0$ as:

$$[u_1, u_2, u_3, \theta](x_i) = \sum_{i=1}^{4} A_i [\delta_1, \delta_2, \delta_3, \delta_4]_{\lambda=-\lambda_i} e^{-\lambda_i x_1}, \tag{30}$$

$$\tau_{11} = \sum_{i=1}^{4} \big[(h_{14} - \lambda_i)\delta_1|_{\lambda=-\lambda_i} + (h_{15} - \lambda_i h_{12})\delta_2|_{\lambda=-\lambda_i} + (h_{16} - \lambda_i h_{13})\delta_3|_{\lambda=-\lambda_i} \\ - \delta_4|_{\lambda=-\lambda_i}\big] A_i e^{-\lambda_i x_1}, \tag{31}$$

$$\tau_{22} = \sum_{i=1}^{4} \big[(h_{24} - \lambda_i h_{21})\delta_1|_{\lambda=-\lambda_i} + (h_{25} - \lambda_i h_{22})\delta_2|_{\lambda=-\lambda_i} + (h_{26} - \lambda_i h_{23})\delta_3|_{\lambda=-\lambda_i} \\ - \beta_2\delta_4|_{\lambda=-\lambda_i}\big] A_i e^{-\lambda_i x_1}, \tag{32}$$

$$\tau_{33} = \sum_{i=1}^{4} \big[(h_{34} - \lambda_i h_{31})\delta_1|_{\lambda=-\lambda_i} + (h_{35} - \lambda_i h_{32})\delta_2|_{\lambda=-\lambda_i} + (h_{36} - \lambda_i h_{33})\delta_3|_{\lambda=-\lambda_i} \\ - \beta_3\delta_4|_{\lambda=-\lambda_i}\big] A_i e^{-\lambda_i x_1}, \tag{33}$$

$$\tau_{23} = \sum_{i=1}^{4} \big[(h_{44} - \lambda_i h_{41})\delta_1|_{\lambda=-\lambda_i} + (h_{45} - \lambda_i h_{42})\delta_2|_{\lambda=-\lambda_i} \\ + (h_{46} - \lambda_i h_{43})\delta_3|_{\lambda=-\lambda_i}\big] A_i e^{-\lambda_i x_1}, \tag{34}$$

$$\tau_{13} = \sum_{i=1}^{4} \big[(h_{54} - \lambda_i h_{51})\delta_1|_{\lambda=-\lambda_i} + (h_{55} - \lambda_i h_{52})\delta_2|_{\lambda=-\lambda_i} + (h_{56} - \lambda_i h_{53})\delta_3|_{\lambda=-\lambda_i}\big] A_i\, e^{-\lambda_i x_1}, \tag{35}$$

$$\tau_{12} = \sum_{i=1}^{4} \big[(h_{64} - \lambda_i h_{61})\delta_1|_{\lambda=-\lambda_i} + (h_{65} - \lambda_i h_{62})\delta_2|_{\lambda=-\lambda_i} + (h_{66} - \lambda_i h_{63})\delta_3|_{\lambda=-\lambda_i}\big] A_i\, e^{-\lambda_i x_1}. \tag{36}$$

6. Boundary conditions

To determine the arbitrary constants A_i, boundary conditions are considered at the surface on the half space $x_1 = 0$ as in Sarkar and Lahiri [13].

Mechanical boundary condition: On the traction, free boundary of the half-space:

$$x_1 = 0, \quad \tau_{11}(0, x_2, x_3, t) = \tau_{22}(0, x_2, x_3, t) = \tau_{33}(0, x_2, x_3, t) = 0. \tag{37}$$

Thermal boundary condition:

$$q_n + \nu\theta = r(0, x_2, x_3, t). \tag{38}$$

Since, $q_n = -\partial\theta/\partial n$, we get $\nu\theta^* - \partial\theta/\partial x_1 = r^*$ at $x_1 = 0$, where q_n is the normal components of the heat flux vector, ν is the Biot's number, $\nu \to 0$ corresponding thermally insulated boundary, $\nu \to +\infty$ corresponding to isothermal boundary. $r(0, x_2, x_3, t)$ is the intensity of the applied heat source.

With the help of Eq. (14), Eqs. (37) and (38) become (omitting '*' for convenience):

$$\tau_{11}(0, x_2, x_3, t) = \tau_{22}(0, x_2, x_3, t) = \tau_{33}(0, x_2, x_3, t) = 0,$$
$$\nu\theta - \frac{\partial\theta}{\partial x_1} = 0, \quad x_1 = 0.$$

7. Numerical analysis

For the purpose of illustrating the problem, we now consider a numerical example for which computational results are presented. Since ω is complex, we take $\omega = \omega_0 + i\varsigma$, for studying the effect of wave propagation, we use the following physical parameters in SI units given in the following.

The numerical constants are given by:

$c_{11} = 16.248$ GPa, $c_{14} = -1.152$ GPa, $c_{25} = 1.608$ GPa, $c_{22} = 11.88$ GPa,
$c_{15} = 0$ GPa, $c_{26} = 1.248$ GPa, $c_{33} = 12.216$ GPa, $c_{16} = 0.561$ GPa,
$c_{32} = 1.032$ GPa, $c_{12} = 1.48$ GPa, $c_{21} = 1.48$ GPa, $c_{34} = 0.672$ GPa,
$c_{13} = 2.4$ GPa, $c_{23} = 1.032$ GPa, $c_{35} = 0.216$ GPa, $c_{41} = -1.152$ GPa,
$c_{24} = 0.912$ GPa, $c_{36} = 0.216$ GPa, $c_{42} = 0.912$ GPa, $c_{51} = 0$ GPa,
$c_{61} = 0.561$ GPa, $c_{43} = -0.672$ GPa, $c_{52} = 1.608$ GPa, $c_{62} = 1.248$ GPa,
$c_{44} = 5.64$ GPa, $c_{53} = 0.216$ GPa, $c_{63} = -0.216$ GPa, $c_{45} = 2.16$ GPa
$c_{54} = 2.16$ GPa, $c_{64} = 0$ GPa, $c_{46} = 0$ GPa, $c_{55} = 5.88$ GPa, $c_{65} = 0$ GPa,
$c_{56} = 0$ GPa, $c_{66} = 6.91$ GPa, $c_E = 0.787$, $\beta_{11} = 7.042$, $\beta_{11} = 7.046$,
$\beta_{11} = 6.09$, $k_{11} = 0.0921$, $k_{11} = 0.0963$, $k_{11} = 0.0917$, $\theta_0 = 293$.

Variations of the stress components and temperature distribution has been graphically shown from Fig. 1 to Fig. 7 for G-N model of type III comparing with G-N model of type II with rotation and without rotation with respect to x_1 for the constant values of $t = 0.5$ and $\omega = 0.2$.

(1) The nature of the stress component τ_{11} is contraction in nature for G-N model of type II and III with rotation $\Omega = 25$ whereas without rotation stress is parallel to the x_1 axis. Stresses for G-N model type II and III are same and parallel in $0 < x_1 < 3$ and then gradually decreasing.

(2) Fig. 2 shows that the stress component τ_{22} is extensive in nature. The maximum value occurs for G-N model of type III with rotation. Stresses are gradually increasing in $0 < x_1 < 0.2$ and then decreasing.

(3) Nature of the stress τ_{33} is same as τ_{22}.

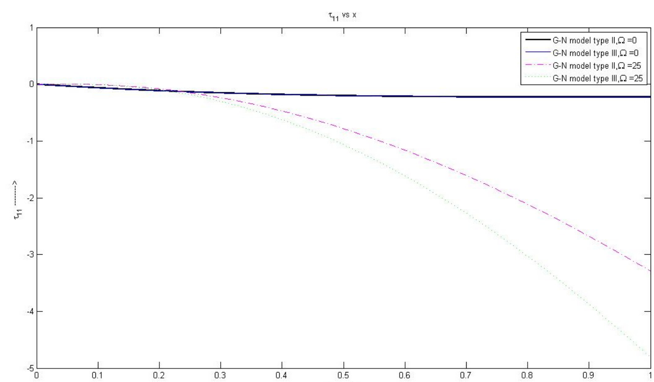

Fig. 1. Variation of τ_{11} with respect to x_1 with rotation and without rotation

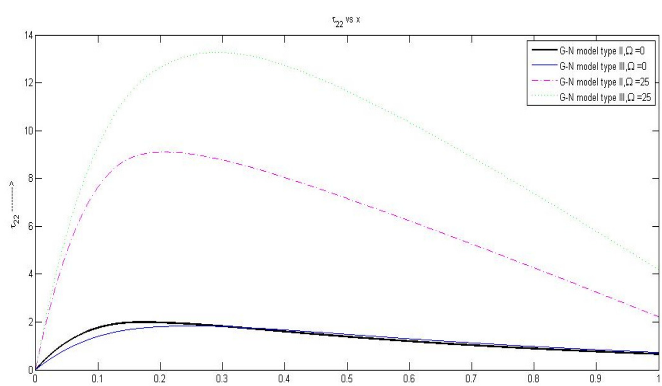

Fig. 2. Variation of τ_{22} with respect to x_1 with rotation and without rotation

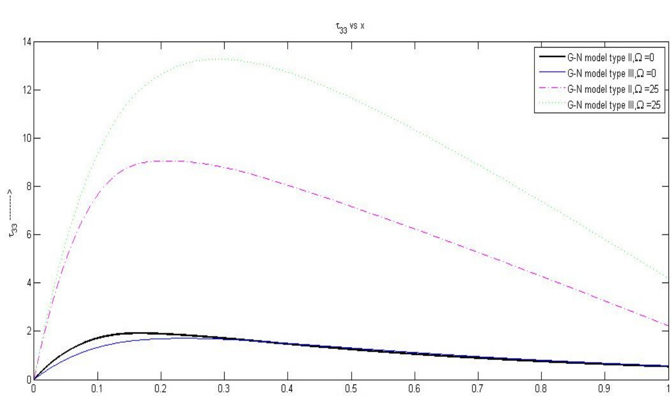

Fig. 3. Variation of τ_{33} with respect to x_1 with rotation and without rotation

(4) In Fig. 4 nature of the stress component τ_{12} is same for G-N model II and III without ritation. Small variation can be seen for the case of with rotation in the range $0 < x_1 < 0.2$. and then stresses are gradually decreasing for all the four cases.

(5) In Fig. 5 nature of τ_{13} is same as τ_{12} for the case of without rotation. And for the case of with rotation the stresses are parallel to the x_1 axis except for a small variation in the range $0.3 < x_1 < 0.5$ in case of G-N model type III.

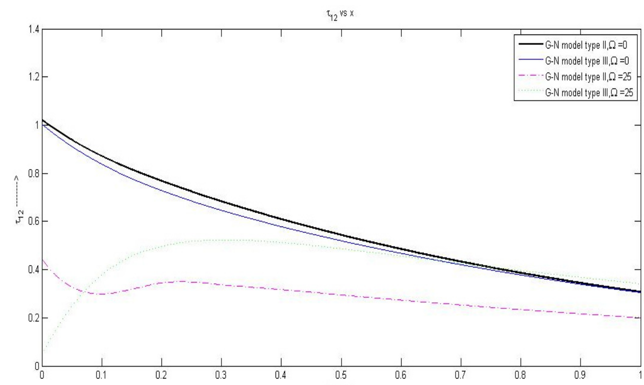

Fig. 4. Variation of τ_{12} with respect to x_1 with rotation and without rotation

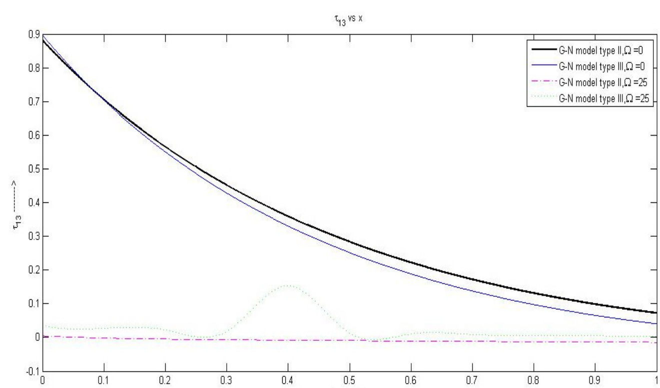

Fig. 5. Variation of τ_{13} with respect to x_1 with rotation and without rotation

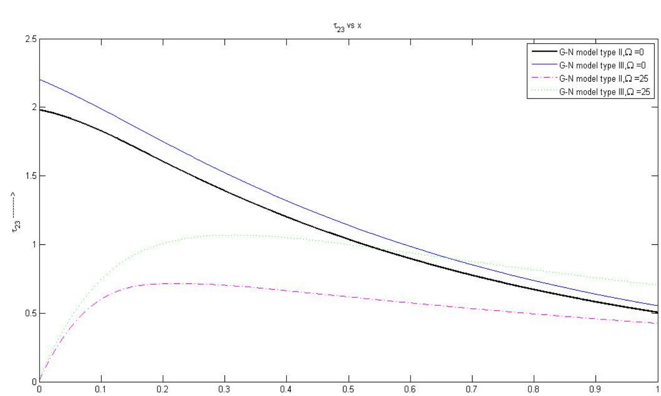

Fig. 6. Variation of τ_{23} with respect to x_1 with rotation and without rotation

(6) Fig. 6 shows that initial value of τ_{23} for G-N model type II and III with rotation is 0 for

$x_1 = 0$ whereas without rotation the initial values are 2 and 2.2 for G-N model II and III respectively. In the range $0 < x_1 < 1$, the stress component τ_{23} monotonically decreases without rotation whereas it increases initially starting from 0, attains a maximum value and then gradually decreases $x_1 > 0.3$ with rotation.

(7) From Fig. 7 we can see that the temperature is gradually increasing in $0 < x_1 < 0.2$ and then maximum value occurs at $x_1 = 0.2$ for the case of G-N model type III with rotation and after that temperature for all the four cases are gradually decreasing.

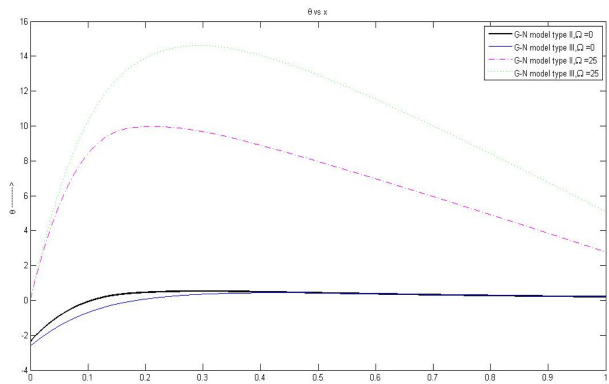

Fig. 7. Variation of θ with respect to x_1 with rotation and without rotation

8. Conclusions

Thermal stresses and temperature on a traction free boundary in a half space for G-N model type II as well as type III due to time dependent heat source shows significant dependence on rotation as evident from the above curves.

Various stress components deviate under rotation from their rotation less counter parts. Certain stress component possesses non zero initial value when subject to rotation whereas for temperature a converse effect has been found.

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Appendix

$$a_{11}=2ia\frac{c_{16}}{c_{11}}+2ib\frac{c_{15}}{c_{11}},\ \ a_{12}=-a^2\frac{c_{66}}{c_{11}}-b^2\frac{c_{55}}{c_{11}}-2ab\frac{c_{56}}{c_{11}}-\omega^2,\ \ a_{21}=\frac{c_{16}}{c_{11}},$$

$$a_{22}=ia\frac{c_{12}+c_{66}}{c_{11}}+ib\frac{c_{14}+c_{56}}{c_{11}},\ \ a_{23}=-a^2\frac{c_{26}}{c_{11}}-b^2\frac{c_{45}}{c_{11}}-ab\frac{c_{46}+c_{25}}{c_{11}},\ \ a_{31}=\frac{c_{15}}{c_{11}},$$

$$a_{32}=ia\frac{c_{14}+c_{56}}{c_{11}}+ib\frac{c_{13}+c_{55}}{c_{11}},\ \ a_{33}=-a^2\frac{c_{46}}{c_{11}}-b^2\frac{c_{35}}{c_{11}}-ab\frac{c_{36}+c_{45}}{c_{11}},$$

$$b_{11}=\frac{c_{16}}{c_{66}},\ \ b_{12}=ia\frac{c_{12}+c_{66}}{c_{66}}+ib\frac{c_{14}+c_{56}}{c_{66}},\ \ b_{13}=-a^2\frac{c_{26}}{c_{66}}-b^2\frac{c_{45}}{c_{66}}-ab\frac{c_{46}+c_{25}}{c_{66}},$$

$$b_{21}=2ia\frac{c_{26}}{c_{66}}+2ib\frac{c_{46}}{c_{66}},\ \ b_{22}=-a^2\frac{c_{22}}{c_{66}}-b^2\frac{c_{44}}{c_{66}}-2ab\frac{c_{24}}{c_{66}}-\frac{c_{11}(\omega^2-\Omega^2)}{c_{66}},$$

$$b_{31}=\frac{c_{56}}{c_{66}},\ \ b_{32}=ia\frac{c_{46}+c_{25}}{c_{66}}+ib\frac{c_{36}+c_{45}}{c_{66}},$$

$$b_{22}=-a^2\frac{c_{24}}{c_{66}}-b^2\frac{c_{34}}{c_{66}}-ab\frac{c_{23}+c_{44}}{c_{66}}+\frac{2\omega\Omega}{c_{66}},$$

$$m_{11}=\frac{c_{15}}{c_{55}},\ \ m_{12}=ia\frac{c_{56}+c_{14}}{c_{55}}+ib\frac{c_{55}+c_{13}}{c_{55}},$$

$$m_{13}=-a^2\frac{c_{46}}{c_{55}}-b^2\frac{c_{35}}{c_{55}}-ab\frac{c_{45}+c_{36}}{c_{55}},\ \ m_{21}=\frac{c_{56}}{c_{55}},$$

$$m_{22}=ia\frac{c_{25}+c_{46}}{c_{55}}+ib\frac{c_{45}+c_{36}}{c_{55}},\ \ m_{23}=-a^2\frac{c_{24}}{c_{55}}-b^2\frac{c_{34}}{c_{55}}-ab\frac{c_{44}+c_{23}}{c_{55}},$$

$$m_{31}=2ia\frac{c_{45}}{c_{55}}+2ib\frac{c_{35}}{c_{55}},\ \ m_{32}=-a^2\frac{c_{44}}{c_{55}}-b^2\frac{c_{33}}{c_{55}}-2ab\frac{c_{34}}{c_{55}}-\frac{c_{11}(\omega^2-\Omega^2)}{c_{55}},$$

$$d_{11}=1+\frac{a_{21}(b_{31}m_{11}-b_{11})}{1-b_{31}m_{21}}+\frac{a_{31}(b_{11}m_{21}-m_{11})}{1-b_{31}m_{21}},$$

$$d_{12}=a_{11}+\frac{a_{21}(b_{31}m_{12}-b_{12})}{1-b_{31}m_{21}}+\frac{a_{31}(b_{12}m_{21}-m_{12})}{1-b_{31}m_{21}},$$

$$d_{13}=a_{12}+\frac{a_{21}(b_{31}m_{13}-b_{13})}{1-b_{31}m_{21}}+\frac{a_{31}(b_{13}m_{21}-m_{13})}{1-b_{31}m_{21}},$$

$$d_{14}=a_{22}+\frac{a_{21}(b_{31}m_{22}-b_{21})}{1-b_{31}m_{21}}+\frac{a_{31}(b_{21}m_{21}-m_{22})}{1-b_{31}m_{21}},$$

$$d_{15} = a_{23} + \frac{a_{21}(b_{31}m_{23} - b_{22})}{1 - b_{31}m_{21}} + \frac{a_{31}(b_{22}m_{21} - m_{23})}{1 - b_{31}m_{21}},$$
$$d_{16} = a_{32} + \frac{a_{21}(b_{31}m_{31} - b_{32})}{1 - b_{31}m_{21}} + \frac{a_{31}(b_{32}m_{21} - m_{31})}{1 - b_{31}m_{21}},$$
$$d_{17} = a_{33} + \frac{a_{21}(b_{31}m_{32} - b_{33})}{1 - b_{31}m_{21}} + \frac{a_{31}(b_{33}m_{21} - m_{32})}{1 - b_{31}m_{21}},$$
$$d_{18} = -\frac{a_{21}(b_{31}m_{33} - b_{34})}{1 - b_{31}m_{21}} - \frac{a_{31}(b_{34}m_{21} - m_{33})}{1 - b_{31}m_{21}},$$
$$k_{11} = -b_{11} - \frac{b_{31}(m_{21}b_{11} - m_{11})}{1 - b_{31}m_{21}}, \quad k_{12} = -b_{21} - \frac{b_{31}(m_{21}b_{12} - m_{12})}{1 - b_{31}m_{21}},$$
$$k_{13} = -b_{13} - \frac{b_{31}(m_{21}b_{13} - m_{13})}{1 - b_{31}m_{21}}, \quad k_{14} = -b_{21} - \frac{b_{31}(m_{21}b_{21} - m_{22})}{1 - b_{31}m_{21}},$$
$$k_{15} = -b_{22} - \frac{b_{31}(m_{21}b_{22} - m_{23})}{1 - b_{31}m_{21}}, \quad k_{16} = -b_{32} - \frac{b_{31}(m_{21}b_{32} - m_{31})}{1 - b_{31}m_{21}},$$
$$k_{17} = -b_{33} - \frac{b_{31}(m_{21}b_{33} - m_{32})}{1 - b_{31}m_{21}}, \quad k_{18} = b_{34} + \frac{b_{31}(m_{21}b_{34} - m_{33})}{1 - b_{31}m_{21}},$$
$$d_{21} = k_{12} - \frac{k_{11}d_{12}}{d_{11}}, \quad d_{22} = k_{13} - \frac{k_{11}d_{13}}{d_{11}}, \quad d_{23} = k_{14} - \frac{k_{11}d_{14}}{d_{11}}, \quad d_{24} = k_{15} - \frac{k_{11}d_{15}}{d_{11}},$$
$$d_{25} = k_{16} - \frac{k_{11}d_{16}}{d_{11}}, \quad d_{26} = k_{17} - \frac{k_{11}d_{17}}{d_{11}}, \quad d_{27} = \frac{k_{11}}{d_{11}},$$
$$d_{31} = \frac{m_{11}d_{12}}{d_{11}} - m_{12} - m_{21}d_{21}, \quad d_{32} = \frac{m_{11}d_{13}}{d_{11}} - m_{13} - m_{21}d_{22},$$
$$d_{33} = \frac{m_{11}d_{14}}{d_{11}} - m_{22} - m_{21}d_{23}, \quad d_{34} = \frac{m_{11}d_{15}}{d_{11}} - m_{23} - m_{21}d_{24},$$
$$d_{35} = \frac{m_{11}d_{16}}{d_{11}} - m_{31} - m_{21}d_{25}, \quad d_{36} = \frac{m_{11}d_{17}}{d_{11}} - m_{32} - m_{21}d_{26},$$
$$d_{37} = -\frac{m_{11}}{d_{11}} + m_{21}d_{27}, \quad d_{38} = \frac{m_{11}d_{18}}{d_{11}} + m_{33} - m_{21}d_{28},$$
$$d_{41} = \frac{\epsilon_0\omega^2}{\epsilon_3 + \epsilon_4\omega}, \quad d_{44} = \frac{i\,a\,\epsilon_1\omega^2}{\epsilon_3 + \epsilon_4\omega}, \quad d_{46} = \frac{i\,b\,\epsilon_2\omega^2}{\epsilon_3 + \epsilon_4\omega},$$
$$d_{48} = \frac{\epsilon_3(a^2 + b^2) + \epsilon_4\omega\left(\frac{k_{22}}{k_{11}}a^2 + \frac{k_{33}}{k_{11}}b^2\right) + \omega^2}{\epsilon_3 + \epsilon_4\omega},$$
$$h_{12} = \frac{c_{16}}{c_{11}}, \quad h_{13} = \frac{c_{15}}{c_{11}}, \quad h_{14} = \frac{i(bc_{15} + ac_{16})}{c_{11}}, \quad h_{15} = \frac{i(bc_{14} + ac_{12})}{c_{11}},$$
$$h_{16} = \frac{i(bc_{13} + ac_{14})}{c_{11}}, \quad h_{21} = \frac{c_{21}}{c_{11}}, \quad h_{22} = \frac{c_{26}}{c_{11}}, \quad h_{23} = \frac{c_{25}}{c_{11}}, \quad h_{24} = \frac{i(bc_{25} + ac_{26})}{c_{11}},$$
$$h_{25} = \frac{i(bc_{24} + ac_{22})}{c_{11}}, \quad h_{26} = \frac{i(bc_{23} + ac_{24})}{c_{11}}, \quad h_{31} = \frac{c_{31}}{c_{11}}, \quad h_{32} = \frac{c_{36}}{c_{11}}, \quad h_{33} = \frac{c_{35}}{c_{11}},$$
$$h_{34} = \frac{i(bc_{35} + ac_{36})}{c_{11}}, \quad h_{35} = \frac{i(bc_{34} + ac_{32})}{c_{11}}, \quad h_{36} = \frac{i(bc_{33} + ac_{34})}{c_{11}}, \quad h_{41} = \frac{c_{41}}{c_{11}},$$
$$h_{42} = \frac{c_{46}}{c_{11}}, \quad h_{43} = \frac{c_{45}}{c_{11}}, \quad h_{44} = \frac{i(bc_{45} + ac_{46})}{c_{11}}, \quad h_{45} = \frac{i(bc_{44} + ac_{42})}{c_{11}},$$
$$h_{46} = \frac{i(bc_{43} + ac_{44})}{c_{11}},$$
$$f_{11} = g_{51} + \lambda\, g_{55} - \lambda^2, \quad f_{12} = g_{52} + \lambda\, g_{56}, \quad f_{13} = g_{53} + \lambda\, g_{57}, \quad f_{14} = g_{54} + \lambda\, g_{58},$$
$$f_{21} = g_{61} + \lambda\, g_{65}, \quad f_{22} = g_{62} + \lambda\, g_{66} - \lambda^2, \quad f_{23} = g_{63} + \lambda\, g_{67}, \quad f_{24} = g_{64} + \lambda\, g_{68},$$

$f_{31} = g_{71} + \lambda\, g_{75},\ \ f_{32} = g_{72} + \lambda\, g_{76},\ \ f_{33} = g_{73} + \lambda\, g_{77} - \lambda^2,\ \ f_{34} = g_{74} + \lambda\, g_{78},$
$f_{41} = \lambda\, g_{85},\ \ f_{42} = g_{82},\ \ f_{43} = g_{83},\ \ f_{44} = g_{84},$
$g_{51} = -\dfrac{d_{13}}{d_{11}},\ \ g_{52} = -\dfrac{d_{15}}{d_{11}},\ \ g_{53} = -\dfrac{d_{17}}{d_{11}},\ \ g_{54} = -\dfrac{d_{18}}{d_{11}},\ \ g_{55} = -\dfrac{d_{12}}{d_{11}},\ \ g_{56} = -\dfrac{d_{14}}{d_{11}},$
$g_{57} = -\dfrac{d_{16}}{d_{11}},\ \ g_{58} = \dfrac{1}{d_{11}},\ \ g_{61} = d_{22},\ \ g_{62} = d_{24},\ \ g_{63} = d_{26},\ \ g_{64} = d_{28},\ \ g_{65} = d_{21},$
$g_{66} = d_{23},\ \ g_{67} = d_{25},\ \ g_{68} = d_{27},\ \ g_{71} = d_{32},\ \ g_{72} = d_{34},\ \ g_{73} = d_{36},\ \ g_{74} = d_{38},$
$g_{75} = d_{31},\ \ g_{76} = d_{33},\ \ g_{77} = d_{35},\ \ g_{78} = d_{37},\ \ g_{81} = 0,\ \ g_{82} = d_{44},\ \ g_{83} = d_{46},$
$g_{84} = d_{48},\ \ g_{85} = d_{41},\ \ g_{86} = 0,\ \ g_{87} = 0,\ \ g_{88} = 0,$
$h_{51} = \dfrac{c_{51}}{c_{11}},\ \ h_{52} = ,\ \ h_{53} = \dfrac{c_{55}}{c_{11}},\ \ h_{54} = \dfrac{i(bc_{55} + ac_{56})}{c_{11}},\ \ h_{55} = \dfrac{i(bc_{54} + ac_{52})}{c_{11}},$
$h_{56} = \dfrac{i(bc_{53} + ac_{54})}{c_{11}},\ \ h_{61} = \dfrac{c_{61}}{c_{11}},\ \ h_{62} = \dfrac{c_{66}}{c_{11}},\ \ h_{63} = \dfrac{c_{65}}{c_{11}},\ \ h_{64} = \dfrac{i(bc_{65} + ac_{66})}{c_{11}},$
$h_{65} = \dfrac{i(bc_{64} + ac_{62})}{c_{11}},\ \ h_{66} = \dfrac{i(bc_{63} + ac_{64})}{c_{11}},$

$$L_{11} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},\quad L_{12} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$L_{21} = \begin{bmatrix} g_{51} & g_{52} & g_{53} & g_{54} \\ g_{61} & g_{62} & g_{63} & g_{64} \\ g_{71} & g_{72} & g_{73} & g_{74} \\ g_{81} & g_{82} & g_{83} & g_{84} \end{bmatrix},\quad L_{22} = \begin{bmatrix} g_{55} & g_{56} & g_{57} & g_{58} \\ g_{65} & g_{66} & g_{67} & g_{68} \\ g_{75} & g_{76} & g_{77} & g_{78} \\ g_{85} & g_{86} & g_{87} & g_{88} \end{bmatrix}.$$