

26. Plane wave propagation in a 3D anisotropic half-space under Green-Naghdi theory II

S. Chakraborty¹, S. C. Mandal², A. K. Das³, N. Sarkar⁴, A. Lahiri⁵

^{1,2,5}Department of Mathematics, Jadavpur University, Kolkata, 700032, India

³Department of Geology, Asutosh College, 92, S. P. Mukherjee Road, Kolkata, 700026, India

⁴Department of Applied Mathematics, University of Calcutta, Kolkata, 700009, India

⁴Corresponding author

E-mail: ¹*csanjukta1977@gmail.com*, ²*scmandal@math.jdvu.ac.in*, ³*akdas.geology@gmail.com*,

⁴*nsarkarindian@gmail.com*, ⁵*lahiriabhijit2000@yahoo.com*

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Abstract. In this paper, the theory of coupled thermoelasticity in three dimension is employed for triclinic half-space, subjected to time dependent heat source on the boundary of the space which is traction free and is considered in the context of Green-Naghdi model of type II (thermoelasticity without energy dissipation) of generalized thermoelasticity. Normal mode analysis is used to the non-dimensional coupled equations. Finally, the resulting equations are written in the form of a vector-matrix differential equation which is then solved by eigenvalue approach. Numerical results for the temperature, thermal stresses, and displacements are presented graphically and analyzed. Mathematical results shown in thermoelastic curves were supplemented by tectonic movements of elastic lithospheric plates.

Keywords: anisotropic solid, Green-Naghdi model II, normal mode analysis, eigenvalue approach, lithostatic pressure and lithospheric plates.

Nomenclature

c_{ijkl}	Material constants
$\beta_{ij} = c_{ijkl}\alpha_{kl}$ ($i, j, k, l = 1, 2, 3$)	Thermal elastic coupling tensor
α_{ij}	Thermal expansion tensor
δ_{ij}	Kronecker delta
u_i	Displacement components
e_{ij}	Strain tensor
τ_{ij}	Stress components
ρ	Mass density
θ_0	Reference temperature
θ	Absolute temperature chosen such that $ \theta - \theta_0/\theta_0 \ll 1$
k_{ij}^*	Green-Naghdi thermal Parameter
C_E	Specific heat at constant strain
c_v	Speed of propagation of isothermal elastic wave
c_1	Dilatation wave velocity

1. Introduction

The classical uncoupled theory of thermoelasticity, predicts two phenomena not consistent with experimental results. The heat conduction equation of this theory i) does not contain any elastic terms, but the fact is that the elastic changes produces heat effects and ii) the heat conduction equation is of parabolic type predicting infinite speeds of propagation for heat waves, which means that the theory classical of thermoelasticity (CT) predicts finite for predominantly elastic disturbances but an infinite speed for predominantly thermal disturbances, which are coupled together. This means that a part of every solution of the equations extends to infinity. To overcome this paradox inherent in the theory of classical uncoupled theory of thermoelasticity,

Biot [1] introduced the theory of coupled thermoelasticity.

To eliminate the second shortcoming Lord and Shulman [2] (L-S model) introduce a theory of generalized thermoelasticity with one relaxation time parameter, by modification of Fourier's law, replaced heat flux and its time derivatives. In L-S model the heat conduction equation is hyperbolic type and is closely connected with the theories of "second sound". This theory was extended for anisotropic body by Dhaliwal and Sherief [3]. The uniqueness of the solutions for this theory was proved under different conditions by Ignaczak [4, 5]. Green and Lindsay [6] (G-L model) modified not only heat conduction equation but also equation of motion in the coupled theory without violating Fourier's law by introducing two relaxation time parameters. Along with the theories, Green and Naghdi [7-9] (G-N model) also proposed another three generalized theories of thermoelasticity by introducing "thermal displacement gradient" among the independent constitutive variables and named as Type I, II and III. Among these models, type I [7] is same as classical heat equation which is based on Fourier's law where the theories are linearized. The type II [9] and type III [8] model permit finite speed of wave propagation. The basic difference of type II from type I and type III is that it does not contain dissipation of thermal energy whereas type III contains dissipation of energy. The type II and Type III are also known as thermoelasticity without energy dissipation (TEWOED) and thermoelasticity with energy dissipation (TEWED). Several investigations relating to TEWOED theory have been studied by Roy Choudhury and Bandyopadhyay [10], Roy Choudhury and Dutta [11], Sharma and Chouhan [12], Chandrasekharaiah and Srinath [13], Sarkar and Lahiri [14] and Bachher et al. [MME15]. The effect of magnetic field in generalized thermoelasticity with internal heat source in half space was investigated by Othman et al. [16]. Pal and Acharya [17] studied the effects of inhomogeneity on the surface waves in anisotropic media. Recently, Pal et al. [18] investigated the wave propagation in an inhomogeneous anisotropic generalized thermoelastic solid. They have also shown by graphically that quasi-P wave, quasi-S wave and thermal wave are discontinuous at certain angle and with a jump as well.

We know that materials in the earth are not homogeneous although their elastic properties vary with depth from one region to another. The variation may be gradual; there are also discontinuities that separate media with different densities and elastic properties. Anisotropy causes the largest variation of seismic velocities and changes in the direction of seismic waves. Achenbach [19] studied reflection and transmission of wave in infinite elastic plate. Several investigations relating to reflection and transmission of elastic waves in anisotropic media have been studied by Nayfeh [20], Pal and Chattopadhyay [21], Chattopadhyay and Rogerson [22], Sharma [23] and Chattopadhyay et al. [24]. Mensch and Rasolofosaon [25] studied the elastic wave velocities in anisotropic media, generalizing Thomson's parameters, δ and γ . Abbas and Othman [26] investigated the propagation of plane waves in fiber-reinforced, anisotropic half-space under effect of hydrostatic initial in L-S model. The analysis of Lamé waves in anisotropic thin plates in generalized thermoelasticity was discussed by Verma [27]. Zhou and Greenhalgh [28], Grechka [29] studied elastic wave group velocities for (generalized) anisotropic medium. The petrophysical study of velocity of waves in anisotropic shale was analyzed by Vernik and Liu [30], Kumar and Gupta [31] investigated the plane wave propagation in anisotropic thermoelastic medium having fractional order derivative and void in the context of three- phase lag model and two phase lag model. Recently Othman et al. [32] studied the effect of rotation on three dimensional generalized thermoelasticity in the context of Green-Naghdi type II for homogeneous isotropic elastic half space.

The thermo-elastic stress behavior and strain calculations due to time-dependent thermal perturbation in homogeneous plates have significant applications in tectonic movements of more homogeneous rigid plates e.g., basaltic oceanic plates subjected to heat sources of hot magmatic plumes of deeper mantle origin. Thus the deduction and modifications of thermo-elastic stresses in proportion to the degree of reheating of the elastic lithosphere are explained in various hotspot thermal models (Zhu and Wiens [33]). The aim of the present research article to study the distribution of stresses, strains and temperature for an anisotropic half space subjected to a time

dependent heat source on the boundary of the space which is traction free in the context of Green-Naghdi model – II. Normal mode analysis technique and eigenvalue approach have been used to solve the problem. Finally, numerical results are presented graphically and analyzed. Some applications have been also given.

2. Basic equations

In the absence of body forces and inner heat sources, the field equations for linear thermoelastic homogeneous anisotropic body in the context of Green-Naghdi model II [9], Othman et al. [32] are as follows:

The equations of motion:

$$\tau_{ij,j} = \rho \ddot{u}_i. \quad (1)$$

Heat- conduction equation of Green- Naghdi theory of type II:

$$k_{ij}^* \theta_{,ij} = \rho c_E \ddot{\theta} + \theta_0 b_{ij} \ddot{u}_{i,j}. \quad (2)$$

The Duhamel-Neumann constitutive equations are:

$$\tau_{ij} = c_{ijkl} e_{kl} - \beta_{ij} \theta \delta_{ij}. \quad (3)$$

Strain – displacement relation:

$$e_{ij} = \frac{1}{2} (u_{i,j} + u_{j,i}), \quad u_{i,j} = \frac{\partial u_i}{\partial x_j}. \quad (4)$$

The comma notation is used for derivatives with respect to space variables x_i and superimposed dot represents time differentiation.

3. Formulation of the problem

We shall consider a triclinic linear thermoelastic anisotropic half-space solid which fills the region $\Omega = \{(x_1, x_2, x_3): 0 \leq x_1 < \infty, -\infty < x_2 < \infty, -\infty < x_3 < \infty\}$ and subjected to time dependent heat sources on the boundary plane to the surface $x_1 = 0$. The body is initially at rest and the surface $x_1 = 0$ is assumed to be traction free.

For three dimensional plane waves in a homogeneous anisotropic elastic medium, the components of displacement vector have the form:

$$u_i = u_i(x_1, x_2, x_3, t), \quad i = 1, 2, 3, \quad (5)$$

where t is the time variable and x_i ($i = 1, 2, 3$) denotes the respective orthogonal Cartesian co-ordinate axes. Using Hooke's law, the stress-strain-temperature relations in a triclinic medium can be written as follows:

$$\begin{aligned} \tau_{11} &= c_{11}e_{11} + c_{12}e_{22} + c_{13}e_{33} + 2(c_{14}e_{23} + c_{15}e_{13} + c_{16}e_{12}) - \beta_{11}\theta, \\ \tau_{22} &= c_{21}e_{11} + c_{22}e_{22} + c_{23}e_{33} + 2(c_{24}e_{23} + c_{25}e_{13} + c_{26}e_{12}) - \beta_{22}\theta, \\ \tau_{33} &= c_{31}e_{11} + c_{32}e_{22} + c_{33}e_{33} + 2(c_{34}e_{23} + c_{35}e_{13} + c_{36}e_{12}) - \beta_{33}\theta, \\ \tau_{23} &= c_{41}e_{11} + c_{42}e_{22} + c_{43}e_{33} + 2(c_{44}e_{23} + c_{45}e_{13} + c_{46}e_{12}), \\ \tau_{13} &= c_{51}e_{11} + c_{52}e_{22} + c_{53}e_{33} + 2(c_{54}e_{23} + c_{55}e_{13} + c_{56}e_{12}), \\ \tau_{12} &= c_{61}e_{11} + c_{62}e_{22} + c_{63}e_{33} + 2(c_{64}e_{23} + c_{65}e_{13} + c_{66}e_{12}). \end{aligned} \quad (6)$$

The equations of motion in absence of body forces and heat sources are as follows:

$$\begin{aligned}\tau_{11,1} + \tau_{12,2} + \tau_{13,3} &= \rho \ddot{u}_1, \\ \tau_{21,1} + \tau_{22,2} + \tau_{23,3} &= \rho \ddot{u}_2, \\ \tau_{31,1} + \tau_{32,2} + \tau_{33,3} &= \rho \ddot{u}_3.\end{aligned}\tag{7}$$

With the help of Eq. (5) and (6), equations of motion (7) become:

$$\begin{aligned}\rho \ddot{u}_1 &= (c_{11}u_{1,11} + c_{66}u_{1,22} + c_{55}u_{1,33}) + 2(c_{16}u_{1,12} + c_{15}u_{1,13} + c_{56}u_{1,23}) \\ &\quad + (c_{16}u_{2,11} + c_{26}u_{2,22} + c_{45}u_{2,33} + (c_{12} + c_{66})u_{2,12} + (c_{14} + c_{56})u_{2,13} \\ &\quad + (c_{46} + c_{25})u_{2,23}) + (c_{15}u_{3,11} + c_{46}u_{3,22} + c_{35}u_{3,33} + (c_{14} + c_{56})u_{3,12} \\ &\quad + (c_{13} + c_{55})u_{3,13} + (c_{36} + c_{45})u_{3,23}) - \beta_{11}\theta_{,1}, \\ \rho \ddot{u}_2 &= (c_{16}u_{1,11} + c_{26}u_{1,22} + c_{45}u_{1,33} + (c_{12} + c_{66})u_{1,12} \\ &\quad + (c_{14} + c_{56})u_{1,13} + (c_{46} + c_{25})u_{1,23}) + (c_{66}u_{2,11} + c_{22}u_{2,22} + c_{44}u_{2,33}) \\ &\quad + 2(c_{26}u_{2,12} + c_{46}u_{2,13} + c_{24}u_{2,23}) + (c_{56}u_{3,11} + c_{24}u_{3,22} + c_{34}u_{3,33} \\ &\quad + (c_{46} + c_{25})u_{3,12} + (c_{36} + c_{45})u_{3,13} + (c_{23} + c_{44})u_{3,23}) - \beta_{22}\theta_{,2}, \\ \rho \ddot{u}_3 &= (c_{15}u_{1,11} + c_{46}u_{1,22} + c_{35}u_{1,33} + (c_{56} + c_{14})u_{1,12} \\ &\quad + (c_{55} + c_{13})u_{1,13} + (c_{45} + c_{36})u_{1,23}) + (c_{56}u_{2,11} + c_{24}u_{2,22} + c_{34}u_{2,33} \\ &\quad + (c_{25} + c_{46})u_{2,12} + (c_{45} + c_{36})u_{2,13} + (c_{44} + c_{23})u_{2,23}) \\ &\quad + (c_{55}u_{3,11} + c_{44}u_{3,22} + c_{33}u_{3,33}) + 2(c_{45}u_{3,12} + c_{35}u_{3,13} + c_{34}u_{3,23}) - \beta_{33}\theta_{,3}.\end{aligned}\tag{8}$$

The generalized heat conduction Eq. (2) is written as:

$$k_{11}^* \theta_{,11} + k_{22}^* \theta_{,22} + k_{33}^* \theta_{,33} = \rho c_E \ddot{\theta} + \theta_0 (b_{11} \ddot{u}_{1,1} + b_{22} \ddot{u}_{2,2} + b_{33} \ddot{u}_{3,3}).\tag{9}$$

To transform the above equations in non-dimensional forms, we introduce the following non-dimensional variables:

$$x'_i = \frac{1}{l} x_i, \quad u'_i = \frac{\rho c_1^2}{lb_{11}\theta_0} u_i, \quad t' = \frac{c_1}{l} t, \quad \theta' = \frac{1}{\theta_0} \theta, \quad \tau'_{ij} = \frac{1}{b_{11}\theta_0} \tau_{ij}, \quad c_1^2 = \frac{c_{11}}{\rho},\tag{10}$$

where l is some standard length.

Using the parameters defined in Eq. (10), the non-dimensional forms of the equations of motion, heat conduction equation and stress components can be obtained as (omitting primes for convenience):

$$\begin{aligned}\ddot{u}_1 &= \left(u_{1,11} + \frac{c_{66}}{c_{11}} u_{1,22} + \frac{c_{55}}{c_{11}} u_{1,33} \right) + 2 \left(\frac{c_{16}}{c_{11}} u_{1,12} + \frac{c_{15}}{c_{11}} u_{1,13} + \frac{c_{56}}{c_{11}} u_{1,23} \right) \\ &\quad + \left(\frac{c_{16}}{c_{11}} u_{2,11} + \frac{c_{26}}{c_{11}} u_{2,22} + \frac{c_{45}}{c_{11}} u_{2,33} \right) + \left(\frac{(c_{12} + c_{66})}{c_{11}} u_{2,12} + \frac{(c_{14} + c_{56})}{c_{11}} u_{2,13} \right. \\ &\quad + \frac{(c_{46} + c_{25})}{c_{11}} u_{2,23} \left. \right) + \left(\frac{c_{15}}{c_{11}} u_{3,11} + \frac{c_{46}}{c_{11}} u_{3,22} + \frac{c_{35}}{c_{11}} u_{3,33} + \frac{(c_{14} + c_{56})}{c_{11}} u_{3,12} \right. \\ &\quad + \frac{(c_{13} + c_{55})}{c_{11}} u_{3,13} + \left. \frac{(c_{36} + c_{45})}{c_{11}} u_{3,23} \right) - \theta_{,1}, \\ \ddot{u}_2 &= \left(\frac{c_{16}}{c_{11}} u_{1,11} + \frac{c_{26}}{c_{11}} u_{1,22} + \frac{c_{45}}{c_{11}} u_{1,33} + \frac{(c_{12} + c_{66})}{c_{11}} u_{1,12} + \frac{(c_{14} + c_{56})}{c_{11}} u_{1,13} \right. \\ &\quad + \left. \frac{(c_{46} + c_{25})}{c_{11}} u_{1,23} \right) + \left(\frac{c_{66}}{c_{11}} u_{2,11} + \frac{c_{22}}{c_{11}} u_{2,22} + \frac{c_{44}}{c_{11}} u_{2,33} \right)\end{aligned}\tag{11}$$

$$\begin{aligned}
 &+ 2 \left(\frac{c_{26}}{c_{11}} u_{2,12} + \frac{c_{46}}{c_{11}} u_{2,13} + \frac{c_{24}}{c_{11}} u_{2,23} \right) + \left(\frac{c_{56}}{c_{11}} u_{3,11} + \frac{c_{24}}{c_{11}} u_{3,22} + \frac{c_{34}}{c_{11}} u_{3,33} \right. \\
 &+ \left. \frac{(c_{46} + c_{25})}{c_{11}} u_{3,12} + \frac{(c_{36} + c_{45})}{c_{11}} u_{3,13} + \frac{(c_{23} + c_{44})}{c_{11}} u_{3,23} \right) - b_2 \theta_{,2}, \\
 \ddot{u}_3 = &\left(\frac{c_{15}}{c_{11}} u_{1,11} + \frac{c_{46}}{c_{11}} u_{1,22} + \frac{c_{35}}{c_{11}} u_{1,33} + \frac{(c_{56} + c_{14})}{c_{11}} u_{1,12} + \frac{(c_{55} + c_{13})}{c_{11}} u_{1,13} \right. \\
 &+ \left. \frac{(c_{45} + c_{36})}{c_{11}} u_{1,23} \right) + \left(\frac{c_{56}}{c_{11}} u_{2,11} + \frac{c_{24}}{c_{11}} u_{2,22} + \frac{c_{34}}{c_{11}} u_{2,33} + \frac{(c_{25} + c_{46})}{c_{11}} u_{2,12} \right. \\
 &+ \left. \frac{(c_{45} + c_{36})}{c_{11}} u_{2,13} + \frac{(c_{44} + c_{23})}{c_{11}} u_{2,23} \right) + \left(\frac{c_{55}}{c_{11}} u_{3,11} + \frac{c_{44}}{c_{11}} u_{3,22} + \frac{c_{33}}{c_{11}} u_{3,33} \right) \\
 &+ 2 \left(\frac{c_{45}}{c_{11}} u_{3,12} + \frac{c_{35}}{c_{11}} u_{3,13} + \frac{c_{34}}{c_{11}} u_{3,23} \right) - \beta_3 \theta_{,3}, \\
 c_T^2 (\theta_{,11} + k_2 \theta_{,22} + k_3 \theta_{,33}) = &\ddot{\theta} + \varepsilon_1 (\ddot{u}_{1,1} + \beta_2 \ddot{u}_{2,2} + \beta_3 \ddot{u}_{3,3}), \tag{12}
 \end{aligned}$$

$$\begin{aligned}
 \tau_{11} = &\frac{1}{c_{11}} [c_{11} e_{11} + c_{12} e_{22} + c_{13} e_{33} + 2(c_{14} e_{23} + c_{15} e_{13} + c_{16} e_{12})] - \theta, \\
 \tau_{22} = &\frac{1}{c_{11}} [c_{21} e_{11} + c_{22} e_{22} + c_{23} e_{33} + 2(c_{24} e_{23} + c_{25} e_{13} + c_{26} e_{12})] - \beta_2 \theta, \\
 \tau_{33} = &\frac{1}{c_{11}} [c_{31} e_{11} + c_{32} e_{22} + c_{33} e_{33} + 2(c_{34} e_{23} + c_{35} e_{13} + c_{36} e_{12})] - \beta_3 \theta, \\
 \tau_{23} = &\frac{1}{c_{11}} [c_{41} e_{11} + c_{42} e_{22} + c_{43} e_{33} + 2(c_{44} e_{23} + c_{45} e_{13} + c_{46} e_{12})], \\
 \tau_{13} = &\frac{1}{c_{11}} [c_{51} e_{11} + c_{52} e_{22} + c_{53} e_{33} + 2(c_{54} e_{23} + c_{55} e_{13} + c_{56} e_{12})], \\
 \tau_{12} = &\frac{1}{c_{11}} [c_{61} e_{11} + c_{62} e_{22} + c_{63} e_{33} + 2(c_{64} e_{23} + c_{65} e_{13} + c_{66} e_{12})],
 \end{aligned} \tag{13}$$

where:

$$c_T^2 = \frac{k_{11}^*}{\rho c_E c_1^2}, \quad \varepsilon_1 = \frac{b_{11}^2 q_0}{r c_{11} C_E}, \quad \varepsilon_r = \varepsilon_1 \beta_{rr}, \quad k_r = \frac{k_{rr}^*}{k_{11}^*}, \quad \beta_r = \frac{\beta_{rr}}{\beta_{11}}, \quad r = 2, 3.$$

4. Normal mode analysis: Formulation of vector–matrix differential equation

For the solution of the Eqs. (11) and (12), the physical variables can be decomposed in terms of normal modes [14] in the following form:

$$[u_1, u_2, u_3, e_{ij}, \theta, \tau_{ij}](x_1, x_2, x_3, t) = [u_1^*, u_2^*, u_3^*, e_{ij}^*, \theta^*, \tau_{ij}^*](x_1) e^{\omega t + i(ax_2 + bx_3)}, \tag{14}$$

where $i = \sqrt{-1}$, ω is the angular frequency and a, b are the wave numbers along x_2 and x_3 directions respectively.

Substituting from Eq. (14) in Eqs. (11)-(13), we obtain (omitting ‘*’ for convenience):

$$u_{1,11} + a_{11} u_{1,1} + a_{12} u_1 + a_{21} u_{2,11} + a_{22} u_{2,1} + a_{23} u_2 + a_{31} u_{3,11} + a_{32} u_{3,1} + a_{33} u_3 - \theta_{,1} = 0, \tag{15}$$

$$b_{11} u_{1,11} + b_{12} u_{1,1} + b_{13} u_1 + u_{2,11} + b_{21} u_{2,1} + b_{22} u_2 + b_{31} u_{2,11} + b_{32} u_{2,1} + b_{33} u_3 - b_{34} \theta = 0, \tag{16}$$

$$m_{11} u_{1,11} + m_{12} u_{1,1} + m_{13} u_1 + m_{21} u_{2,11} + m_{22} u_{2,1} + m_{23} u_2 + u_{3,11} + m_{31} u_{3,1} + m_{32} u_3 - m_{33} \theta = 0, \tag{17}$$

$$c_T^2 \theta_{,11} - \varepsilon_1 \omega^2 u_{1,1} - ia \omega^2 \varepsilon_2 u_2 - ib \omega^2 \varepsilon_3 u_3 - [c_T^2 (k_2 a^2 + k_3 b^2) - \omega^2] \theta = 0, \tag{18}$$

$$\tau_{11} = u_{1,1} + h_{12}u_{2,1} + h_{13}u_{3,1} + h_{14}u_1 + h_{15}u_2 + h_{16}u_3 - \theta, \quad (19)$$

$$\tau_{22} = h_{21}u_{1,1} + h_{22}u_{2,1} + h_{23}u_{3,1} + h_{24}u_1 + h_{25}u_2 + h_{26}u_3 - \beta_2\theta, \quad (20)$$

$$\tau_{33} = h_{31}u_{1,1} + h_{32}u_{2,1} + h_{33}u_{3,1} + h_{34}u_1 + h_{35}u_2 + h_{36}u_3 - \beta_3\theta, \quad (21)$$

$$\tau_{23} = h_{41}u_{1,1} + h_{42}u_{2,1} + h_{43}u_{3,1} + h_{44}u_1 + h_{45}u_2 + h_{46}u_3, \quad (22)$$

$$\tau_{13} = h_{51}u_{1,1} + h_{52}u_{2,1} + h_{53}u_{3,1} + h_{54}u_1 + h_{55}u_2 + h_{56}u_3, \quad (23)$$

$$\tau_{12} = h_{61}u_{1,1} + h_{62}u_{2,1} + h_{63}u_{3,1} + h_{64}u_1 + h_{65}u_2 + h_{66}u_3, \quad (24)$$

where a_{ij} , b_{ij} , m_{ij} and h_{ij} ($i, j = 1, 2, 3$) are given in the Appendix 1.

Eqs. (15)-(18) can be written in vector-matrix differential equation [14, 34] as follows:

$$\frac{d\underline{v}}{dx_1} = \underline{A}\underline{v}, \quad (25)$$

where $\underline{v}(x_1) = [u_1 u_2 u_3 \theta u_{1,1} u_{2,1} u_{3,1} \theta_{,1}]^T(x_1)$, $\underline{A} = \begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{bmatrix}$.

In the coefficient matrix \underline{A} of order 8, L_{11} is the null matrix of order 4, L_{12} is the identity matrix of order 4 and the matrices L_{21} , L_{22} are given in the Appendix 1.

5. Solution of the vector-matrix differential equation: Eigenvalue approach

For the solution of the vector-matrix differential Eq. (25), we apply the method of eigenvalue approach as in Santra et al. [35]. The characteristic equation of matrix \underline{A} is given by:

$$|A - \lambda I| = 0. \quad (26)$$

The roots (eigenvalues of the matrix \underline{A}) of the characteristic Eq. (26) are of the form $\lambda = \pm\lambda_i$ ($i = 1, 2, 3, 4$).

The eigenvector \underline{X}_λ corresponding to the eigenvalue λ can be calculated as:

$$\underline{X}_\lambda = [\delta_1 \delta_2 \delta_3 \delta_4 \lambda\delta_1 \lambda\delta_2 \lambda\delta_3 \lambda\delta_4]^T, \quad (27)$$

where:

$$\delta_1 = (f_{24}f_{13} - f_{14}f_{23})(f_{22}f_{33} - f_{32}f_{23}) - (f_{34}f_{23} - f_{24}f_{33})(f_{12}f_{23} - f_{22}f_{13}),$$

$$\delta_2 = (f_{34}f_{23} - f_{24}f_{33})(f_{11}f_{23} - f_{21}f_{13}) - (f_{24}f_{13} - f_{14}f_{23})(f_{21}f_{33} - f_{31}f_{23}),$$

$$\delta_3 = (f_{12}f_{21} - f_{11}f_{22})(f_{21}f_{34} - f_{31}f_{24}) - (f_{22}f_{31} - f_{21}f_{32})(f_{11}f_{24} - f_{14}f_{21}),$$

$$\delta_4 = (f_{11}f_{23} - f_{21}f_{13})(f_{22}f_{33} - f_{32}f_{23}) - (f_{12}f_{23} - f_{22}f_{13})(f_{21}f_{33} - f_{31}f_{23}),$$

and f_{ij} ($i, j = 1, 2, 3$) are given in the Appendix 1.

From Eq. (27), we can calculate the eigenvalue \underline{X}_i [$i = 1$ (1) 8] corresponding to the eigenvalue $\lambda = \lambda_i$ [$i = 1$ (1) 8]. For our further reference, we use the following notations:

$$\underline{X}_j = \begin{cases} \underline{X}_{\lambda=\lambda_{i+1}/2}, & i = 1 \text{ (2)}, \\ \underline{X}_{\lambda=-\lambda_i/2}, & i = 2 \text{ (2) } 8. \end{cases} \quad (28)$$

As in Lahiri et al. [36], the general solution of Eq. (25) which is regular as $x_1 \rightarrow +\infty$ can be written as:

$$\underline{v}(x_1) = \sum_{i=1}^4 A_i X_{2i} e^{-\lambda_i x_1}, \quad x_1 \geq 0, \quad (29)$$

where the terms containing exponential of growing nature in the space variables x_1 has been discarded due to the regularity condition of the solution at $+\infty$ and the arbitrary constants A_i are to be determined from the boundary conditions of the problem.

Thus, the field variables can be written from Eq. (29) for $x_1 \geq 0$ as:

$$[u_1, u_2, u_3, \theta](x_1) = \sum_{i=1}^4 A_i [\delta_1, \delta_2, \delta_3, \delta_4]_{\lambda=-\lambda_i} e^{-\lambda_i x_1}, \quad (30)$$

$$\tau_{11} = \sum_{i=1}^4 \left[(h_{14} - \lambda_i) \delta_1|_{\lambda=-\lambda_i} + (h_{15} - \lambda_i h_{12}) \delta_2|_{\lambda=-\lambda_i} + (h_{16} - \lambda_i h_{13}) \delta_3|_{\lambda=-\lambda_i} - \delta_4|_{\lambda=-\lambda_i} \right] A_i e^{-\lambda_i x_i}, \quad (31)$$

$$\tau_{22} = \sum_{i=1}^4 \left[(h_{24} - \lambda_i h_{21}) \delta_1|_{\lambda=-\lambda_i} + (h_{25} - \lambda_i h_{22}) \delta_2|_{\lambda=-\lambda_i} + (h_{26} - \lambda_i h_{23}) \delta_3|_{\lambda=-\lambda_i} - \beta_2 \delta_4|_{\lambda=-\lambda_i} \right] A_i e^{-\lambda_i x_i}, \quad (32)$$

$$\tau_{33} = \sum_{i=1}^4 \left[(h_{34} - \lambda_i h_{31}) \delta_1|_{\lambda=-\lambda_i} + (h_{35} - \lambda_i h_{32}) \delta_2|_{\lambda=-\lambda_i} + (h_{36} - \lambda_i h_{33}) \delta_3|_{\lambda=-\lambda_i} - \beta_3 \delta_4|_{\lambda=-\lambda_i} \right] A_i e^{-\lambda_i x_i}, \quad (33)$$

$$\tau_{23} = \sum_{i=1}^4 \left[(h_{44} - \lambda_i h_{41}) \delta_1|_{\lambda=-\lambda_i} + (h_{45} - \lambda_i h_{42}) \delta_2|_{\lambda=-\lambda_i} + (h_{46} - \lambda_i h_{43}) \delta_3|_{\lambda=-\lambda_i} \right] A_i e^{-\lambda_i x_i}, \quad (34)$$

$$\tau_{13} = \sum_{i=1}^4 \left[(h_{54} - \lambda_i h_{51}) \delta_1|_{\lambda=-\lambda_i} + (h_{55} - \lambda_i h_{52}) \delta_2|_{\lambda=-\lambda_i} + (h_{56} - \lambda_i h_{53}) \delta_3|_{\lambda=-\lambda_i} \right] A_i e^{-\lambda_i x_i}, \quad (35)$$

$$\tau_{12} = \sum_{i=1}^4 \left[(h_{64} - \lambda_i h_{61}) \delta_1|_{\lambda=-\lambda_i} + (h_{65} - \lambda_i h_{62}) \delta_2|_{\lambda=-\lambda_i} + (h_{66} - \lambda_i h_{63}) \delta_3|_{\lambda=-\lambda_i} \right] A_i e^{-\lambda_i x_i}. \quad (36)$$

The simplified form of Eqs. (31)-(36) can be written as:

$$\begin{aligned} \tau_{11} &= A_1 R_{11}(x_1) + A_2 R_{12}(x_1) + A_3 R_{13}(x_1) + A_4 R_{14}(x_1), \\ \tau_{22} &= A_1 R_{21}(x_1) + A_2 R_{22}(x_1) + A_3 R_{23}(x_1) + A_4 R_{24}(x_1), \\ \tau_{33} &= A_1 R_{31}(x_1) + A_2 R_{32}(x_1) + A_3 R_{33}(x_1) + A_4 R_{34}(x_1), \\ \tau_{23} &= A_1 R_{41}(x_1) + A_2 R_{42}(x_1) + A_3 R_{43}(x_1) + A_4 R_{44}(x_1), \\ \tau_{13} &= A_1 R_{51}(x_1) + A_2 R_{52}(x_1) + A_3 R_{53}(x_1) + A_4 R_{54}(x_1), \\ \tau_{12} &= A_1 R_{61}(x_1) + A_2 R_{62}(x_1) + A_3 R_{63}(x_1) + A_4 R_{64}(x_1), \end{aligned} \quad (37)$$

where $R_{ij} = R_{ij}(x_1)$, $[i = 1 (1) 6, j = 1 (1) 4]$ are given in the Appendix 2.

6. Boundary conditions

In order to determine the arbitrary constants A_i , we need to consider the boundary conditions at the surface $x_1 = 0$. The boundary conditions are taken as follows [14]:

I. Mechanical Boundary Condition: the boundary of the half-space $x_1 = 0$ has no traction everywhere i.e.:

$$\tau_{11}(0, x_2, x_3, t) = \tau_{22}(0, x_2, x_3, t) = \tau_{33}(0, x_2, x_3, t) = 0. \quad (38)$$

II. The thermal boundary condition is taken as:

$$q_n + \gamma\theta = r(0, x_2, x_3, t), \tag{39}$$

where q_n denotes the normal components of the heat flux vector, γ is the Biot's number, $\gamma \rightarrow 0$ corresponding thermally insulated boundary and $\gamma \rightarrow +\infty$ represents isothermal boundary condition. The function $r(0, x_2, x_3, t)$ represents the intensity of the applied heat sources. With the help of Eq. (14), Eqs. (38) and (39) become (omitting '*' for convenience):

$$\tau_{11}(0, x_2, x_3, t) = \tau_{22}(0, x_2, x_3, t) = \tau_{33}(0, x_2, x_3, t) = 0, \tag{40}$$

$$\gamma\theta - \frac{d\theta}{dx_1} = r^*, \quad x_1 = 0. \tag{41}$$

Substituting from the boundary Eq. (40) and (41) into the Eqs. (30)-(33), we get following simultaneous equations satisfy by A_1, A_2, A_3 and A_4 :

$$\sum_{i=1}^4 A_i R_{1i}(0) = 0, \quad \sum_{i=1}^4 A_i R_{2i}(0) = 0, \quad \sum_{i=1}^4 A_i R_{3i}(0) = 0, \quad \sum_{i=1}^4 A_i R_{7i}(0) = r^*. \tag{42}$$

Solving the above systems of four Eq. (42), we get: $A_i = \Delta_i/\Delta$ ($i = 1, 2, 3, 4$), where Δ_i and Δ are given in Appendix 2.

7. Numerical example and discussion

In view of illustrating the theoretical results obtained, we now present some numerical results. Since ω is complex, we take $\omega = \omega_0 + i\zeta$. The numerical constants are given in Table 1.

Table 1. The numerical constants

Constants	Values	Constants	Values
c_{11}	16.248 GPa	c_{22}	11.88 GPa
c_{33}	12.216 GPa	c_{12}	1.48 GPa
c_{13}	2.4 GPa	c_{14}	-1.152 GPa
c_{15}	0 GPa	c_{16}	-0.561 GPa
c_{23}	1.032 GPa	c_{24}	0.912 GPa
c_{25}	1.608 GPa	c_{26}	1.248 GPa
c_{34}	-0.672 GPa	c_{35}	0.216 GPa
c_{36}	-0.216 GPa	c_{44}	5.64 GPa
c_{45}	2.16 GPa	c_{46}	0 GPa
c_{55}	5.88 GPa	c_{56}	0 GPa
c_{66}	6.91 GPa	ρ	2.4 kgm ⁻¹
ϵ_1	0.0221	ϵ_2	0.0143
ϵ_3	0.0174	k_{11}^*	1.13×10 ² Wm ⁻¹ deg ⁻¹
k_{22}^*	1.17×10 ² Wm ⁻¹ deg ⁻¹	k_{33}^*	1.21×10 ² Wm ⁻¹ deg ⁻¹

In order to study the characteristic of stresses, strains and temperature, we have drawn several graphs for different values of the space variable x_1 , time t , r^* and ω , we conclude the following observations:

1) In Fig. 1 and 2 we see the variation of dimensionless stress components, for fixed values of $\omega = 4$, $r^* = 20$ and $t = 0.8$ verses space variable x_1 .

- The nature of normal and shearing stresses are almost same with respect to wave propagation.
- The numerical values of τ_{11} , τ_{22} and τ_{33} gradually increase within the region $0 \leq x_1 \leq 0.3$, then gradually decrease and finally vanish as x_1 increases further.
- The numerical value of τ_{11} is much higher than τ_{22} and τ_{33} near the stress free boundary $x_1 = 0$.

• The non-dimension shearing stresses τ_{12} , τ_{13} and τ_{23} start from non-zero value and continuously decrease and finally tends to zero as x_1 increases.

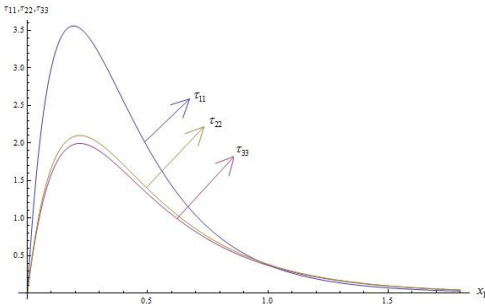


Fig. 1. Distribution of Stresses for $t = 0.8$ and $\omega = 4$ versus x_1

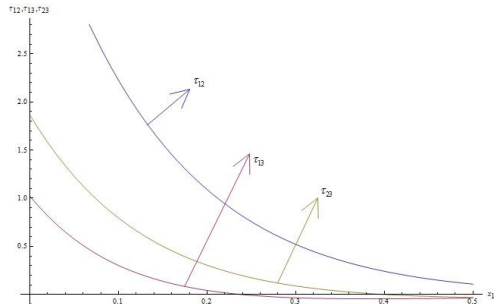


Fig. 2. Distribution of Stresses for $t = 0.8$ and $\omega = 4$ versus x_1

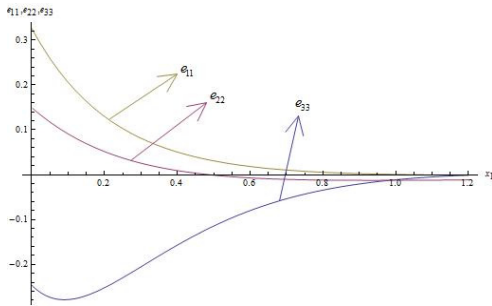


Fig. 3. Distribution of Shearing Strains for $t = 0.3$ and $\omega = 3$ versus x_1

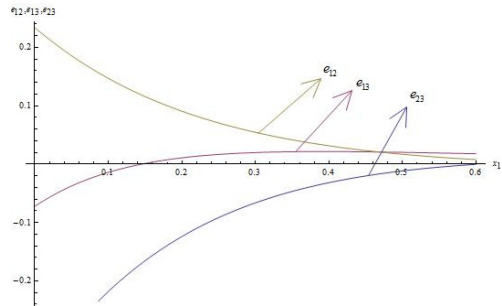


Fig. 4. Distribution of Shearing Strains for $t = 0.3$ and $\omega = 3$ versus x_1

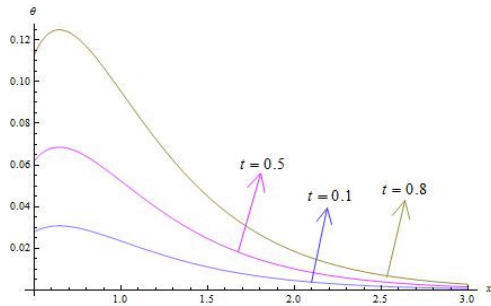


Fig. 5. Distribution of temperature for $\omega = 2$ and $x_2 = x_3 = 0.5$ versus x_1

2) In Fig. 3 and 4 we notice the variation of dimensionless strain components for fixed values of $\omega = 3$, $r^* = 20$ and $t = 0.6$ versus space variable x_1 .

• Absolute values of strain components gradually decrease as space variable x_1 increases and finally vanish.

- The strain components e_{11} , e_{22} and e_{12} are extensive whereas e_{33} and e_{23} are compressive.
- The characteristic of e_{11} , e_{22} and e_{12} are same with respect to wave propagation.
- The non-dimensional shearing strain components e_{13} is compressive within the region $0 \leq x_1 \leq 0.15$ and extensive beyond $x_1 \geq 0.15$ and finally vanish.

3) Fig. 5 represents the variation of non-dimensional temperature θ for fixed time, when $\omega = 2$, $r^* = 20$ and $x_2 = x_3 = 0.5$ versus x_1 .

- Temperature gradually decreases as space variable x_1 increases for all time.

- For fixed x_1 , θ gradually increases as time t increases.

4) Figs. 6-9 represent the variations of stresses and strains for fixed time when $r^* = 20$ and $x_1 = x_2 = x_3 = 0.5$ versus $\omega (\leq 1)$.

- From these figures it is clear that τ_{22} and τ_{12} (normal and shearing stresses), e_{22} and e_{13} (normal and shearing strains) are discontinuous in the neighborhood of $\omega = 0.6$ for $t = 0.1$, $t = 0.5$ and $t = 0.8$.

- τ_{22} is extensive except in the region $0.58 \leq \omega \leq 0.64$ for all $t = 0.1, t = 0.5$ and $t = 0.8$.

- τ_{12} is compressive except in the region $0.57 \leq \omega \leq 0.59$ for all $t = 0.1, 0.5$ and 0.8 .

- From Fig. 8, it is clear that e_{22} is always positive within the region $0 \leq \omega \leq 1$ although it has a discontinuity at $\omega = 0.59$.

- Fig. 9 depicted that e_{13} is always negative within the region $0 \leq \omega \leq 1$ although it has a discontinuity at $\omega = 0.59$.

- $\tau_{22}, \tau_{12}, e_{22}$ and e_{13} are maximum (absolute value) for $t = 0.8$ within the region $0 \leq \omega \leq 1$.

- Nature of discontinuities of $\tau_{11}, \tau_{33}, \tau_{23}$ and τ_{13} are almost same as τ_{22} and τ_{12} .

- The discontinuities of other strain components and e_{11}, e_{33}, e_{12} and e_{23} are also same as e_{22} and e_{13} .

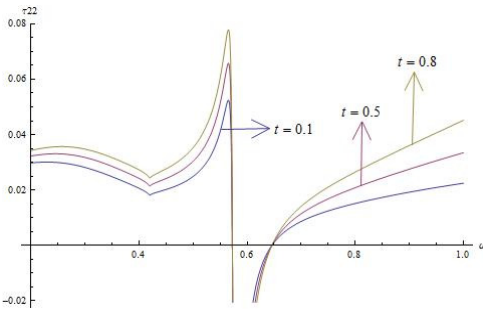


Fig. 6. Distribution of τ_{22} for fixed $x_1 = x_2 = x_3 = 0.5$ and $r^* = 20$ for different values of ω

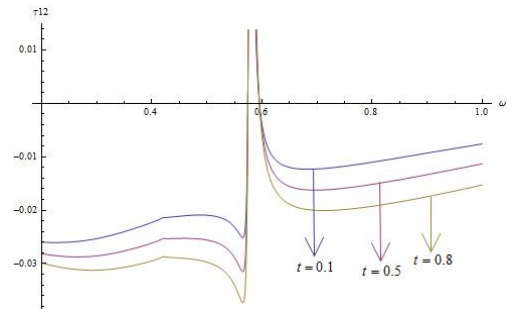


Fig. 7. Distribution of τ_{12} for fixed $x_1 = x_2 = x_3 = 0.5$ and $r^* = 20$ for different values of ω

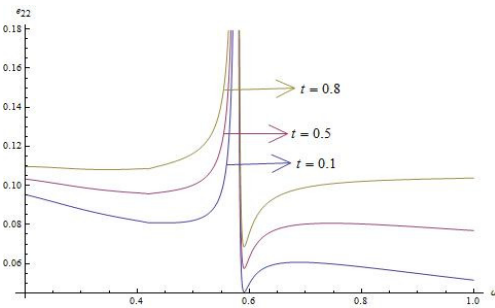


Fig. 8. Distribution of e_{22} for fixed $x_1 = x_2 = x_3 = 0.5$ and $r^* = 20$ for different values of ω

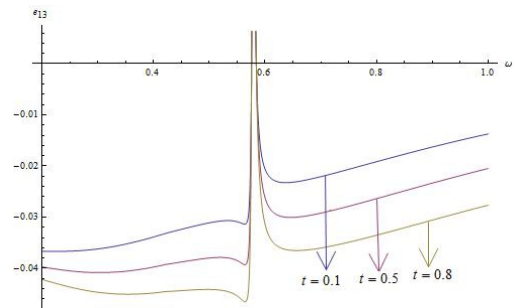


Fig. 9. Distribution of e_{13} for fixed $x_1 = x_2 = x_3 = 0.5$ and $r^* = 20$ for different values of ω

5) Figs. 10-12 depict the distribution of stresses τ_{11}, τ_{12} and τ_{23} for $0 < \omega \leq 1$ and time t in the fixed plane $x_1 = 0.3$ when $r^* = 200$.

- Fig. 10 shows that normal stress τ_{11} is discontinuous within $0.59 \leq \omega \leq 0.61$ for all values of time t ($0 < t < 0.3$). Numerical values of τ_{11} slowly increases with time for fixed ω . On the other hand for fixed time t , τ_{11} changes rapidly when $0.61 \leq \omega \leq 1.0$ for $0 < t \leq 0.3$.

- From Fig. 11 it is clear that shearing stress τ_{12} is compressive for $0 \leq \omega \leq 1.0$ and $0 < t < 0.3$ except at the point of discontinuity $0.58 < \omega < 0.59$. The characteristic of τ_{12} is same for all time when ω is fixed and vice versa.

- Fig. 12 represents the distribution of the shearing stress τ_{23} which is extensive for

$0 < \omega \leq 1$ and $0 < t < 0.3$. It can be noticed that τ_{23} is discontinuous within $0.58 < \omega < 0.60$. τ_{23} gradually decreases for fixed time t when $0.2 < \omega < 0.57$. Beyond the discontinuity region for fixed ω , τ_{23} steadily increases as time t increases. The numerical value of τ_{23} gradually decreases for fixed time as ω increases.

- The nature of τ_{13} is same as τ_{23} .
- The nature of wave propagation of τ_{22} and τ_{33} are same as τ_{11} .

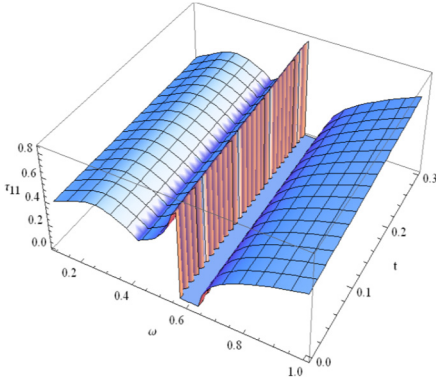


Fig. 10. Distribution of τ_{11} for fixed $x_1 = 0.3$ and different values of ω, t

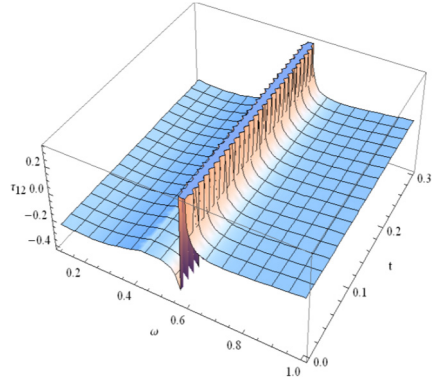


Fig. 11. Distribution of τ_{12} for fixed $x_1 = 0.3$ and different values of ω, t

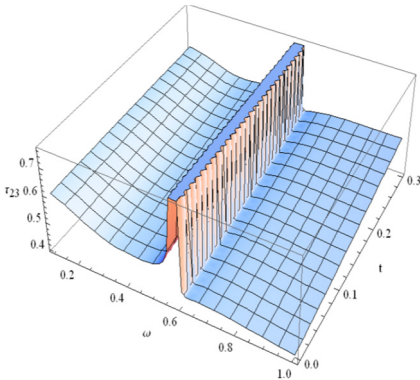


Fig. 12. Distribution of τ_{23} for fixed $x_1 = 0.3$ and different values of ω, t

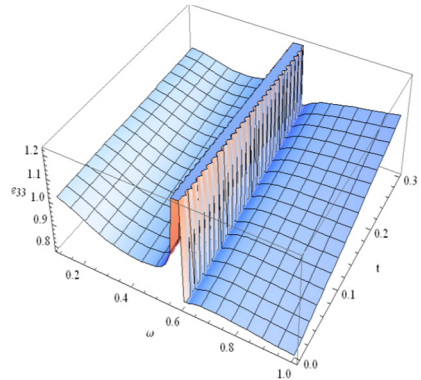


Fig. 13. Distribution of e_{33} for fixed $x_1 = 0.3$ and different values of ω, t

6) Figs. 13-16 represent distribution of strains e_{33} , e_{12} , e_{23} and e_{13} for $\omega (\leq 1)$ and time t in the fixed plane $x_1 = 0.3$ when $r^* = 200$.

• Fig. 13 shows that the normal strain components e_{33} is discontinuous for $0.58 < \omega < 0.60$ and $0 < t \leq 0.3$. Also we see that e_{33} gradually decreases for fixed time when $0.2 < \omega < 0.56$. For fixed ω , e_{33} steadily increases as time t ($0 < t \leq 0.3$) increases.

• Fig. 14 display the distribution of strain components e_{12} which is discontinuous for $0.58 < \omega < 0.60$ and $0 < t \leq 0.3$. For fixed time t , e_{12} gradually decreases within the region $0.2 < \omega \leq 0.40$ and then slowly increases in $0.4 < \omega \leq 0.58$. After the location of the discontinuity, e_{12} gradually increases as ω increases. Nothing significant changes can be noticed for fixed ω when time t changes.

• Fig. 15 depicts that the strain component e_{23} is discontinuous when $0.56 < \omega < 0.60$ for all time $0 < t \leq 0.3$. Also we can notice that e_{23} is positive like e_{33} for $0 < \omega \leq 1$ and $0 < t \leq 0.3$. For fixed ω , e_{23} increases as time t increases.

• Fig. 16 represents the distribution of strain component e_{13} which is discontinuous within the

region $0.58 < \omega < 0.60$ for all time $0 < t \leq 0.3$.

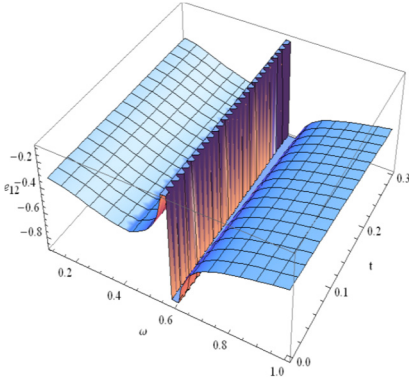


Fig. 14. Distribution of e_{12} for fixed $x_1 = 0.3$ and different values of ω, t

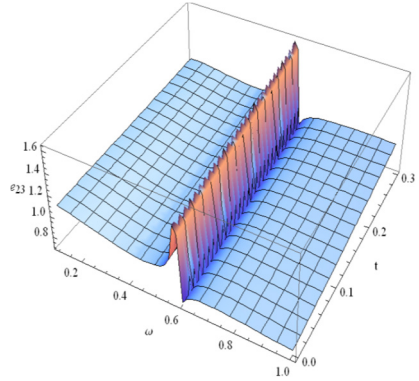


Fig. 15. Distribution of e_{23} for fixed $x_1 = 0.3$ and different values of ω, t

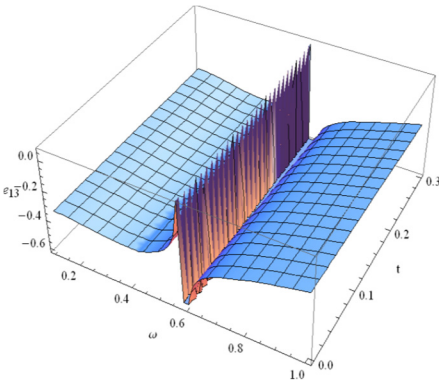


Fig. 16. Distribution of e_{13} for fixed $x_1 = 0.3$ and different values of ω, t

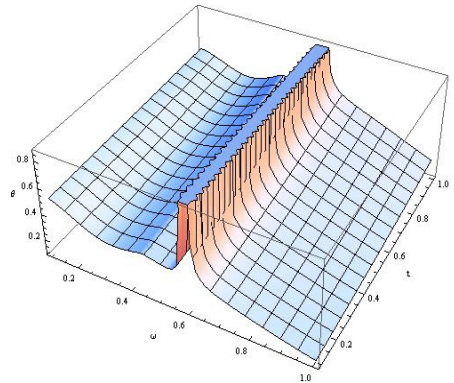


Fig. 17. Distribution of θ for fixed $x_1 = x_2 = x_3 = 0.8, r^* = 200$ and different values of ω, t

7) Fig. 17 displays the distribution of temperature θ for different values of ω and time t . For fixed time, θ gradually decreases for $0 < \omega < 0.56$. Temperature is maximum for all time $0.56 < \omega < 0.60$ and then gradually decreases. For fixed ω , θ slowly increases as time t increases.

8) Figs. 18-20 are drawn to show the distribution of normal and shearing stresses for $0 < \omega \leq 10$ and $0 < t \leq 0.5$ in the fixed plane $x_1 = 0.3$ when $r^* = 200$.

- From Fig. 18 it is clear that significant changes occur within $3.5 < \omega < 7.5$ and $2.5 \leq t \leq 0.4$. For fixed ω , τ_{11} gradually increases as time t ($0.25 \leq t \leq 0.45$) increases and gradually decreases for fixed time when $0 < \omega < 10$.

- Fig. 19 shows that shearing stress τ_{12} gradually increases for fixed ω ($2 \leq \omega \leq 10$) when t ($0 \leq t \leq 0.4$). It can be noted that τ_{12} is maximum when $\omega = 10$ and $t = 0.4$.

- Fig. 20 depicts that for fixed time, τ_{13} gradually decreases as ω increases which is more prominent for $t = 0.4$ and $0 < \omega < 8$. After attaining the minimum value τ_{13} gradually increases for $8 < \omega < 10$ and $0.35 \leq t \leq 0.4$.

- Nature of τ_{22} and τ_{33} are almost same as τ_{11} .

9) Figs. 21-24 show the distribution of strains in the fixed plane $x_1 = 0.3$ when $r^* = 200$ for different values of ω and t .

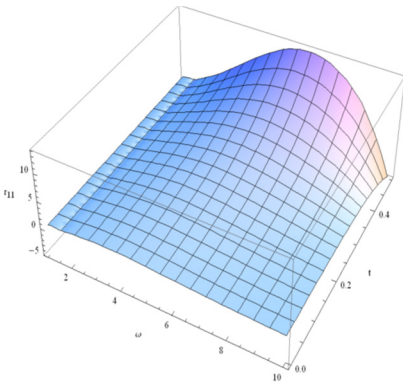


Fig. 18. Distribution of τ_{11} for fixed $x_1 = 0.3$ and different values of ω, t

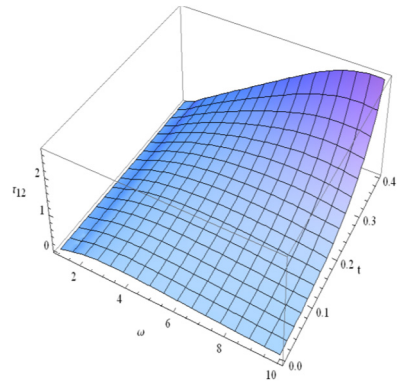


Fig. 19. Distribution of τ_{12} for fixed $x_1 = 0.3$ and different values of ω, t

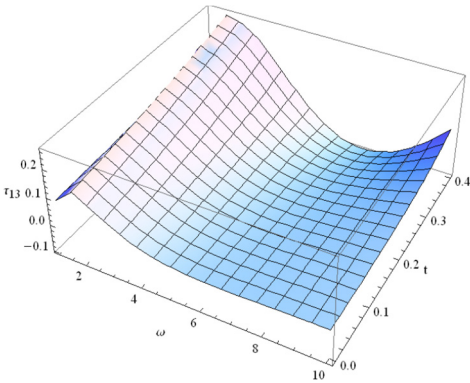


Fig. 20. Distribution of τ_{13} for fixed $x_1 = 0.3$ and different values of ω, t

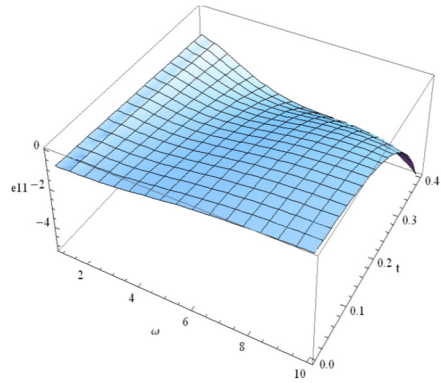


Fig. 21. Distribution of e_{11} for fixed $x_1 = 0.3$ and different values of ω, t

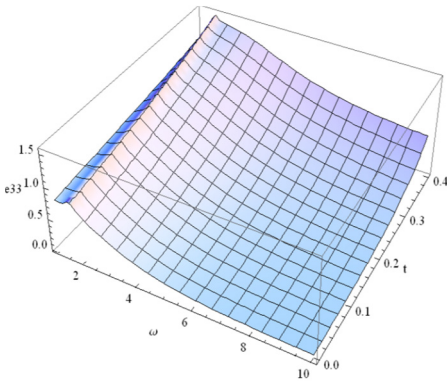


Fig. 22. Distribution of e_{33} for fixed $x_1 = 0.3$ and different values of ω, t

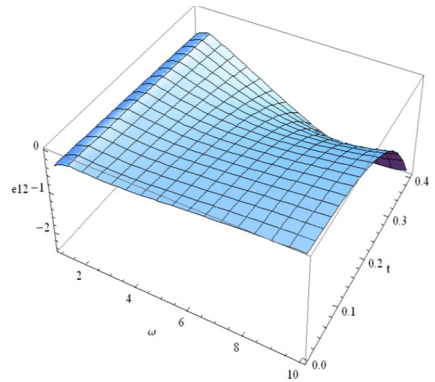


Fig. 23. Distribution of e_{12} for fixed $x_1 = 0.3$ and different values of ω, t

• Figs. 21 and 22 represent the distributions of the normal strain components. We notice that e_{11} is negative for all $0 < \omega \leq 10$ and $0 < t < 0.4$ while e_{33} is positive. Numerical value of e_{11} gradually tends to zero when ω tends to 10 and t tends to 0.4. On the other hand e_{33} becomes zero when $\omega = 10$ and $t = 0.4$.

• Fig. 23 depicts the distribution of the shearing strain e_{12} . Characteristic of e_{12} is almost same as e_{11} .

• Fig. 24 shows that for fixed ω , numerical value of e_{13} gradually increasing as time t increases. For fixed time t , e_{13} slowly increases as ω increases.

10) Fig. 25 displays the temperature distribution for $2 \leq \omega \leq 10$ and $0 < t \leq 0.4$. Temperature attains its maximum value when $\omega = 6$ and $t = 0.4$. Temperature increases slowly when $2 \leq \omega \leq 6$ and $0.15 \leq t \leq 0.4$ then gradually decreases.

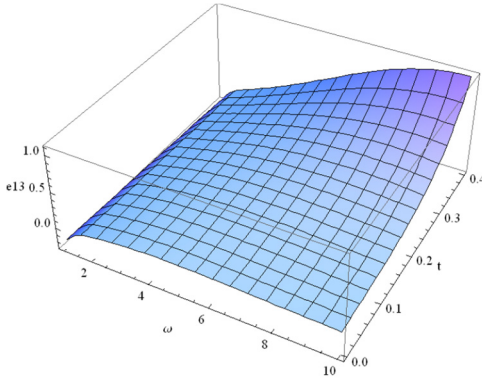


Fig. 24. Distribution of e_{13} for fixed $x_1 = 0.3$ and different values of ω, t

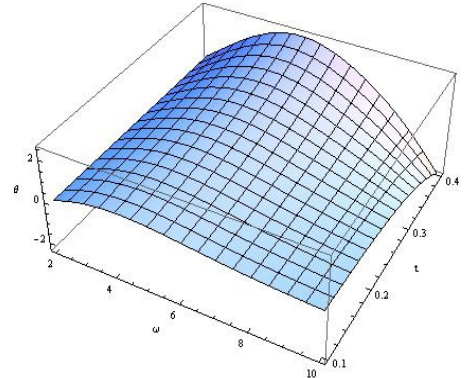


Fig. 25. Distribution of θ for fixed $x_1 = x_2 = x_3 = 0.5, \tau^* = 200$ and different values of ω, t

8. Applications and uncertainties

Subsurface rocks are generally filled with cracks and pore spaces saturated with one or more fluids such as oils, water and gasses which can influence the mechanical behavior of rocks. Thermal expansion of existing fluids due to thermo-poroelastic stress field causes gradual increase of fluid pressure in the intergranular spaces. If the fluid pressure (P_f) exceeds lithostatic pressure (P_l) by more than the tensile strength of the rock (usually small) the rock is likely to burst due to hydraulic fracturing and in the process fluid will migrate along the cracks developed (Fig. 26). The migration of oil and gas can be accentuated through these cracks into suitable structural traps producing voluminous oil and gas field in the rocks.

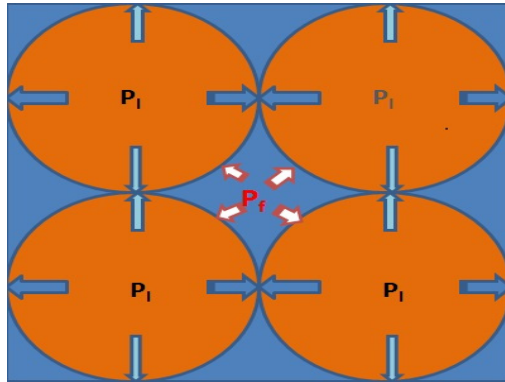


Fig. 26. Schematic diagram showing developments of hydraulic fracturing in matrix of rocks.
 P_l = Lithostatic Pressure and P_f = Fluid Pressure

According to linear elastic fracturing model three distinct modes of fracturing can be generated depending on the three different types of loading (Irwin [37]). In Mode-I, the tensile forces are applied normal to the crack plane so that the crack planes are pulled apart enabling the crack more and more open (Fig. 27(a)). In Mode-II, the cracked body is loaded by shear forces acting parallel to the crack surfaces, which slide over each other in the applied direction (Fig. 27(b)). In Mode-III

the cracked body is also loaded by shear forces but parallel to the crack front and the fracture planes slide over each other out of the plane direction (Fig. 27(c)). In case of Mode-III, a large angle of sliding of rock blocks or plates may indicate a relative rotation of plates, resulting a significant amount of evolution of energy. A distinct discontinuity in the thermo-elastic curves shown whenever the ω value of considerable amount rotated clockwise or anticlockwise (Fig. 27(c)) possibly indicates movements of plate with significant release of energy.

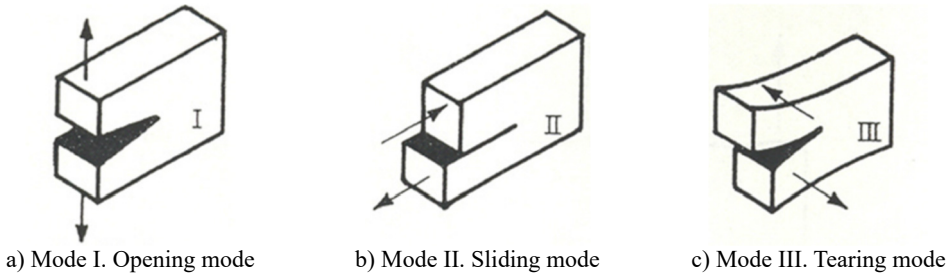


Fig. 27. Three types of fractures modes (Anderson [45])

Thus, lithospheric plates, though rigid, must certainly undergo small elastic strains showing time-dependent deformations or creep, even in very low levels of differential stress. Thus, the time-dependent heat sources appear to produce significant stresses in the lithospheric plates resulting intraplate tectonics (Turcotte [38]). It has been appreciated that the midplate swells are the results of thermal perturbations of intra-plate hot spot plumes as the plates move over the deep mantle thermal anomaly (Wilson [39]). The major uncertainties appear to be the variations of minimum depth of reheating with corresponding assumptions of the magnitude of the conductive heating in the elastic lithosphere. But the magnitudes of the calculated temperature perturbations and the related thermo-elastic stress accumulation are significant as shown in the mathematical calculations. Shallow reheating depths and larger heat flow anomalies obviously increase the temperature perturbations within the lithosphere. The effect of higher heat flow may be partially compensated in case of deeper reheating depths. Thus, the application of thermo-elastic stress analyses is more suitable for relatively thinner and homogeneous younger oceanic lithosphere. The geographical distributions of thermo-elastic stresses and their orientations are highly significant for the tectonic evolution of the plates and can be measured by earthquake focal mechanism.

Several thermo-elastic stress models applied to hotspot reheating show considerably higher deviatoric stresses for both slow and fast moving plates. Although the thermal stress fields orientations appear to be complex in pattern a horizontal extension at shallow depths and horizontal compressions near the bottom of the elastic lithosphere have been appreciated. The thermal stress configuration depends on the absolute plate velocity. Thus, fast moving plates do not get enough time for conducting heat into the poorly conducting elastic plates and as a result a large deviatoric thermal stress generates far downstream hotspot chain. While a slow moving plate appears to get enough time to interact with the underlying hotspot and creates a more circular and larger thermal field closer to the hotspot location. Obviously, the area of the thermal stress perturbation is much larger for a fast moving plate and has little or no effect to the earthquakes related to hotspots e.g., most of the earthquakes in Hawaii Island (Zhu and Wiens [33], Koyanagi et al. [40], Ando [41]). However, in case of slow moving plates the effect of radial thermal stresses closer to the hotspot swells have convincingly shown direct relation to the shallow focus earthquakes (Stewart and Helmerger [42], Nishenko and Kafka [43], Bergman [44]). The focal depths of these earthquakes often mismatch to the predicted depths derived from varied thermal stress models. Thermoelastic stress structures studied in different hotspot swells show an extension perpendicular to the swell axis in shallow depths. The passage of magma through the

upper part of the lithosphere is thought to be guided by this extensional stress direction. In case of moderate to slow movements of plates this tensional stresses appear to accumulate considerably long distances downstream from the hotspot location. Volcanic rejuvenation along island arcs may thus result long after the passage of hotspots. During this long time period any kind of changes in the direction of plates or rotation of plate motion, as discussed, may explain the significant changes in the initial hotspot chains as noted a 65 degrees change in direction of hotspot chain in Hawaii Island. To confirm the pattern of maximum thermal stress orientations more thermal stress modeling are appreciated near hotspots in association with earthquakes. Thus, mathematical calculations related to thermo-elastic behavior of isotropic plates subjected to temperature perturbations and various boundary conditions may show certain pathways to solve many queries about the plate-tectonic processes and associated heat sources.

9. Concluding remarks

In this paper we derive the exact analytical expression for non-dimensional stresses, strains and temperature of a homogeneous anisotropic half space medium of most general symmetry (triclinic) of thermoelasticity without energy dissipation (G-N model II) is considered. The time dependent heat source is applied at the boundary of the half space. We observe the following conclusions based on the above analysis.

- 1) The time dependent heat source has a significant effect on the physical quantities.
- 2) The maximum changes occur near the heated region.
- 3) From Figs. 1-5, it is clear that the numerical values of physical quantities converge to zero as x_1 increases.
- 4) It is clear from Figs. 6-17 that the changes in the values of ω causes discontinuities of the physical quantities in the neighborhood of $0.56 < \omega < 0.62$.
- 5) Figs. 18-25 depicts that the changes in the values of time t and $\omega (>1)$ cause significant changes in all the physical quantities.

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Appendix

A1.

$$\begin{aligned}
 a_{11} &= 2ia \frac{c_{16}}{c_{11}} + 2ib \frac{c_{15}}{c_{11}}, & a_{12} &= -a^2 \frac{c_{66}}{c_{11}} - b^2 \frac{c_{55}}{c_{11}} - 2ab \frac{c_{55}}{c_{11}} - \omega^2, \\
 a_{21} &= \frac{c_{16}}{c_{11}}, & a_{22} &= ia \frac{c_{12} + c_{66}}{c_{11}} + ib \frac{c_{14} + c_{56}}{c_{11}}, & a_{23} &= -a^2 \frac{c_{26}}{c_{11}} - b^2 \frac{c_{45}}{c_{11}} - ab \frac{c_{46} + c_{25}}{c_{11}}, \\
 a_{31} &= \frac{c_{15}}{c_{11}}, & a_{32} &= ia \frac{c_{14} + c_{56}}{c_{11}} + ib \frac{c_{13} + c_{55}}{c_{11}}, & a_{33} &= -a^2 \frac{c_{46}}{c_{11}} - b^2 \frac{c_{35}}{c_{11}} - ab \frac{c_{36} + c_{45}}{c_{11}}, \\
 b_{11} &= \frac{c_{16}}{c_{66}}, & b_{12} &= ia \frac{c_{12} + c_{66}}{c_{66}} + ib \frac{c_{14} + c_{56}}{c_{66}}, & b_{13} &= -a^2 \frac{c_{26}}{c_{66}} - b^2 \frac{c_{45}}{c_{66}} - ab \frac{c_{46} + c_{25}}{c_{66}}, \\
 b_{21} &= 2ia \frac{c_{26}}{c_{66}} + 2ib \frac{c_{46}}{c_{66}}, & b_{22} &= -a^2 \frac{c_{22}}{c_{66}} - b^2 \frac{c_{44}}{c_{66}} - 2ab \frac{c_{24}}{c_{66}} - \frac{c_{11}}{c_{66}} \omega^2, & b_{31} &= \frac{c_{56}}{c_{66}}, \\
 b_{32} &= ia \frac{c_{46} + c_{25}}{c_{66}} + ib \frac{c_{36} + c_{45}}{c_{66}}, & b_{33} &= -a^2 \frac{c_{24}}{c_{66}} - b^2 \frac{c_{34}}{c_{66}} - ab \frac{c_{23} + c_{44}}{c_{66}}, & b_{34} &= \frac{ia\beta_2 c_{11}}{c_{66}}, \\
 m_{11} &= \frac{c_{15}}{c_{55}}, & m_{12} &= ia \frac{c_{56} + c_{14}}{c_{55}} + ib \frac{c_{55} + c_{13}}{c_{55}}, & m_{13} &= -a^2 \frac{c_{46}}{c_{55}} - b^2 \frac{c_{35}}{c_{55}} - ab \frac{c_{45} + c_{36}}{c_{55}}, \\
 m_{21} &= \frac{c_{56}}{c_{55}}, & m_{22} &= ia \frac{c_{25} + c_{46}}{c_{55}} + ib \frac{c_{45} + c_{36}}{c_{55}}, & m_{23} &= -a^2 \frac{c_{24}}{c_{55}} - b^2 \frac{c_{34}}{c_{55}} - ab \frac{c_{44} + c_{23}}{c_{55}}, \\
 m_{31} &= 2ia \frac{c_{45}}{c_{55}} + 2ib \frac{c_{35}}{c_{55}}, & m_{32} &= -a^2 \frac{c_{44}}{c_{55}} - b^2 \frac{c_{33}}{c_{55}} - 2ab \frac{c_{34}}{c_{55}} - \frac{c_{11}}{c_{55}} \omega^2, & m_{33} &= \frac{ib\beta_3 c_{11}}{c_{55}}, \\
 h_{12} &= \frac{c_{16}}{c_{11}}, & h_{13} &= \frac{c_{15}}{c_{11}}, & h_{14} &= \frac{i(bc_{15} + ac_{16})}{c_{11}}, & h_{15} &= \frac{i(bc_{14} + ac_{12})}{c_{11}}, \\
 h_{16} &= \frac{i(bc_{13} + ac_{14})}{c_{11}}, & h_{21} &= \frac{c_{21}}{c_{11}}, & h_{22} &= \frac{c_{26}}{c_{11}}, & h_{23} &= \frac{c_{25}}{c_{11}}, & h_{24} &= \frac{i(bc_{25} + ac_{26})}{c_{11}}, \\
 h_{25} &= \frac{i(bc_{24} + ac_{22})}{c_{11}}, & h_{26} &= \frac{i(bc_{23} + ac_{24})}{c_{11}}, & h_{31} &= \frac{c_{31}}{c_{11}}, & h_{32} &= \frac{c_{36}}{c_{11}}, & h_{33} &= \frac{c_{35}}{c_{11}}, \\
 h_{34} &= \frac{i(bc_{35} + ac_{26})}{c_{11}}, & h_{35} &= \frac{i(bc_{34} + ac_{32})}{c_{11}}, & h_{36} &= \frac{i(bc_{33} + ac_{34})}{c_{11}}, & h_{41} &= \frac{c_{41}}{c_{11}},
 \end{aligned}$$

$$\begin{aligned}
 h_{42} &= \frac{c_{46}}{c_{11}}, & h_{43} &= \frac{c_{45}}{c_{11}}, & h_{44} &= \frac{i(bc_{45} + ac_{46})}{c_{11}}, & h_{45} &= \frac{i(bc_{44} + ac_{42})}{c_{11}}, \\
 h_{46} &= \frac{i(bc_{43} + ac_{44})}{c_{11}}, & h_{51} &= \frac{c_{51}}{c_{11}}, & h_{52} &= \frac{c_{56}}{c_{11}}, & h_{53} &= \frac{c_{55}}{c_{11}}, & h_{54} &= \frac{i(bc_{55} + ac_{56})}{c_{11}}, \\
 h_{55} &= \frac{i(bc_{54} + ac_{52})}{c_{11}}, & h_{56} &= \frac{i(bc_{53} + ac_{54})}{c_{11}}, & h_{61} &= \frac{c_{61}}{c_{11}}, & h_{62} &= \frac{c_{66}}{c_{11}}, \\
 h_{63} &= \frac{c_{65}}{c_{11}}, & h_{64} &= \frac{i(bc_{65} + ac_{66})}{c_{11}}, & h_{65} &= \frac{i(bc_{64} + ac_{62})}{c_{11}}, & h_{66} &= \frac{i(bc_{63} + ac_{64})}{c_{11}}, \\
 d_{11} &= 1 + \frac{a_{21}(b_{31}m_{11} - b_{11})}{1 - b_{31}m_{21}} + \frac{a_{31}(m_{21}b_{11} - m_{11})}{1 - b_{31}m_{21}}, \\
 d_{12} &= a_{11} + \frac{a_{21}(b_{31}m_{12} - b_{12})}{1 - b_{31}m_{21}} + \frac{a_{31}(m_{21}b_{12} - m_{12})}{1 - b_{31}m_{21}}, \\
 d_{13} &= a_{12} + \frac{a_{21}(b_{31}m_{13} - b_{13})}{1 - b_{31}m_{21}} + \frac{a_{31}(m_{21}b_{13} - m_{13})}{1 - b_{31}m_{21}}, \\
 d_{14} &= a_{22} + \frac{a_{21}(b_{31}m_{22} - b_{21})}{1 - b_{31}m_{21}} + \frac{a_{31}(m_{21}b_{21} - m_{22})}{1 - b_{31}m_{21}}, \\
 d_{15} &= a_{23} + \frac{a_{21}(b_{31}m_{23} - b_{22})}{1 - b_{31}m_{21}} + \frac{a_{31}(m_{21}b_{22} - m_{23})}{1 - b_{31}m_{21}}, \\
 d_{16} &= a_{32} + \frac{a_{21}(b_{31}m_{31} - b_{32})}{1 - b_{31}m_{21}} + \frac{a_{31}(m_{21}b_{32} - m_{31})}{1 - b_{31}m_{21}}, \\
 d_{17} &= a_{33} + \frac{a_{21}(b_{31}m_{32} - b_{33})}{1 - b_{31}m_{21}} + \frac{a_{31}(m_{21}b_{33} - m_{31})}{1 - b_{31}m_{21}}, \\
 d_{18} &= -\frac{a_{21}(b_{31}m_{33} - b_{34})}{1 - b_{31}m_{21}} + \frac{a_{31}(m_{33} - m_{21}b_{34})}{1 - b_{31}m_{21}}, \\
 d_{21} &= \left(-b_{12} - \frac{b_{31}(m_{21}b_{12} - m_{12})}{1 - b_{31}m_{21}}\right) - \frac{d_{12}}{d_{11}} \left(-b_{11} - \frac{b_{31}(m_{21}b_{11} - m_{11})}{1 - b_{31}m_{21}}\right), \\
 d_{22} &= \left(-b_{13} - \frac{b_{31}(m_{21}b_{13} - m_{13})}{1 - b_{31}m_{21}}\right) - \frac{d_{13}}{d_{11}} \left(-b_{11} - \frac{b_{31}(m_{21}b_{11} - m_{11})}{1 - b_{31}m_{21}}\right), \\
 d_{23} &= \left(-b_{21} - \frac{b_{31}(m_{21}b_{21} - m_{22})}{1 - b_{31}m_{21}}\right) - \frac{d_{14}}{d_{11}} \left(-b_{11} - \frac{b_{31}(m_{21}b_{11} - m_{11})}{1 - b_{31}m_{21}}\right), \\
 d_{24} &= \left(-b_{22} - \frac{b_{31}(m_{21}b_{22} - m_{23})}{1 - b_{31}m_{21}}\right) - \frac{d_{15}}{d_{11}} \left(-b_{11} - \frac{b_{31}(m_{21}b_{11} - m_{11})}{1 - b_{31}m_{21}}\right), \\
 d_{25} &= \left(-b_{32} - \frac{b_{31}(m_{21}b_{32} - m_{31})}{1 - b_{31}m_{21}}\right) - \frac{d_{16}}{d_{11}} \left(-b_{11} - \frac{b_{31}(m_{21}b_{11} - m_{11})}{1 - b_{31}m_{21}}\right), \\
 d_{27} &= \frac{1}{d_{11}} \left(-b_{11} - \frac{b_{31}(m_{21}b_{11} - m_{11})}{1 - b_{31}m_{21}}\right), & d_{31} &= \frac{m_{11}d_{12}}{d_{11}} - m_{12} - m_{21}d_{21}, \\
 d_{32} &= \frac{m_{11}d_{13}}{d_{11}} - m_{13} - m_{21}d_{22}, & d_{33} &= \frac{m_{11}d_{14}}{d_{11}} - m_{22} - m_{21}d_{23}, \\
 d_{34} &= \frac{m_{11}d_{15}}{d_{11}} - m_{23} - m_{21}d_{24}, & d_{35} &= \frac{m_{11}d_{16}}{d_{11}} - m_{31} - m_{21}d_{25}, \\
 d_{36} &= \frac{m_{11}d_{17}}{d_{11}} - m_{32} - m_{21}d_{26}, & d_{37} &= -\left(\frac{m_{11}\beta_1}{d_{11}} + m_{21}d_{27}\right), \\
 d_{38} &= \frac{m_{11}d_{18}}{d_{11}} + m_{33} - m_{21}d_{28}, & d_{41} &= \frac{e_1\omega^2}{C_T^2}, & d_{42} &= 0, & d_{43} &= 0, & d_{44} &= \frac{iae_2\omega^2}{C_T^2}, \\
 d_{45} &= 0, & d_{46} &= \frac{ibe_3\omega^2}{C_T^2}, & d_{47} &= 0, & d_{48} &= \frac{C_T^2(k_2a^2 + k_3b^2) + \omega^2}{C_T^2},
 \end{aligned}$$

$$\begin{aligned}
 g_{51} &= -\frac{d_{13}}{d_{11}}, & g_{52} &= -\frac{d_{15}}{d_{11}}, & g_{53} &= -\frac{d_{17}}{d_{11}}, & g_{54} &= -\frac{d_{18}}{d_{11}}, & g_{55} &= -\frac{d_{12}}{d_{11}}, & g_{56} &= -\frac{d_{14}}{d_{11}}, \\
 g_{57} &= -\frac{d_{16}}{d_{11}}, & g_{58} &= \frac{1}{d_{11}}, & g_{61} &= d_{22}, & g_{62} &= d_{24}, & g_{63} &= d_{26}, & g_{64} &= d_{28}, \\
 g_{65} &= d_{21}, & g_{66} &= d_{23}, & g_{67} &= d_{25}, & g_{68} &= d_{27}, & g_{71} &= d_{32}, & g_{72} &= d_{34}, & g_{73} &= d_{36}, \\
 g_{74} &= d_{38}, & g_{75} &= d_{31}, & g_{76} &= d_{33}, & g_{77} &= d_{35}, & g_{78} &= d_{37}, & g_{81} &= 0, & g_{82} &= d_{44}, \\
 g_{83} &= d_{46}, & g_{84} &= d_{48}, & g_{85} &= d_{41}, & g_{86} &= g_{87} = g_{88} = 0, \\
 f_{11} &= g_{51} + \lambda g_{55} - \lambda^2, & f_{12} &= g_{52} + \lambda g_{56}, & f_{13} &= g_{53} + \lambda g_{57}, & f_{14} &= g_{54} + \lambda g_{58}, \\
 f_{21} &= g_{61} + \lambda g_{65}, & f_{22} &= g_{62} + \lambda g_{66} - \lambda^2, & f_{23} &= g_{63} + \lambda g_{67}, & f_{24} &= g_{64} + \lambda g_{68}, \\
 f_{31} &= g_{71} + \lambda g_{65}, & f_{32} &= g_{72} + \lambda g_{76}, & f_{33} &= g_{73} + \lambda g_{77} - \lambda^2, & f_{34} &= g_{74} + \lambda g_{78}, \\
 f_{41} &= \lambda g_{85}, & f_{42} &= g_{82}, & f_{43} &= g_{83}, & f_{44} &= g_{84}, \\
 L_{21} &= \begin{pmatrix} g_{51} & g_{52} & g_{53} & g_{54} \\ g_{61} & g_{62} & g_{63} & g_{64} \\ g_{71} & g_{72} & g_{73} & g_{74} \\ g_{81} & g_{82} & g_{83} & g_{84} \end{pmatrix}, & L_{22} &= \begin{pmatrix} g_{55} & g_{56} & g_{57} & g_{58} \\ g_{65} & g_{66} & g_{67} & g_{68} \\ g_{75} & g_{76} & g_{77} & g_{78} \\ g_{85} & g_{86} & g_{87} & g_{88} \end{pmatrix}.
 \end{aligned}$$

A2.

$$\begin{aligned}
 R_{11}(x_1) &= [(h_{14} - \lambda_1)\{(\delta_1)_{\lambda=-\lambda_1} + (h_{15} - \lambda_1 h_{12})\{(\delta_2)_{\lambda=-\lambda_1}\} \\
 &\quad + (h_{16} - \lambda_1 h_{13})\{(\delta_3)_{\lambda=-\lambda_1}\} - \{(\delta_4)_{\lambda=-\lambda_1}\}]e^{-\lambda_1 x_1}, \\
 R_{12}(x_1) &= [(h_{14} - \lambda_2)\{(\delta_1)_{\lambda=-\lambda_2} + (h_{15} - \lambda_2 h_{12})\{(\delta_2)_{\lambda=-\lambda_2}\} \\
 &\quad + (h_{16} - \lambda_2 h_{13})\{(\delta_3)_{\lambda=-\lambda_2}\} - \{(\delta_4)_{\lambda=-\lambda_2}\}]e^{-\lambda_2 x_1}, \\
 R_{13}(x_1) &= [(h_{14} - \lambda_3)\{(\delta_1)_{\lambda=-\lambda_3} + (h_{15} - \lambda_3 h_{12})\{(\delta_2)_{\lambda=-\lambda_3}\} \\
 &\quad + (h_{16} - \lambda_3 h_{13})\{(\delta_3)_{\lambda=-\lambda_3}\} - \{(\delta_4)_{\lambda=-\lambda_3}\}]e^{-\lambda_3 x_1}, \\
 R_{14}(x_1) &= [(h_{14} - \lambda_4)\{(\delta_1)_{\lambda=-\lambda_4} + (h_{15} - \lambda_4 h_{12})\{(\delta_2)_{\lambda=-\lambda_4}\} \\
 &\quad + (h_{16} - \lambda_4 h_{13})\{(\delta_3)_{\lambda=-\lambda_4}\} - \{(\delta_4)_{\lambda=-\lambda_4}\}]e^{-\lambda_4 x_1}, \\
 R_{21}(x_1) &= [(h_{24} - \lambda_1 h_{21})\{(\delta_1)_{\lambda=-\lambda_1} + (h_{25} - \lambda_1 h_{22})\{(\delta_2)_{\lambda=-\lambda_1}\} \\
 &\quad + (h_{26} - \lambda_1 h_{23})\{(\delta_3)_{\lambda=-\lambda_1}\} - \beta_2\{(\delta_4)_{\lambda=-\lambda_1}\}]e^{-\lambda_1 x_1}, \\
 R_{22}(x_1) &= [(h_{24} - \lambda_2 h_{21})\{(\delta_1)_{\lambda=-\lambda_2} + (h_{25} - \lambda_2 h_{22})\{(\delta_2)_{\lambda=-\lambda_2}\} \\
 &\quad + (h_{26} - \lambda_2 h_{23})\{(\delta_3)_{\lambda=-\lambda_2}\} - \beta_2\{(\delta_4)_{\lambda=-\lambda_2}\}]e^{-\lambda_2 x_1}, \\
 R_{23}(x_1) &= [(h_{24} - \lambda_3 h_{21})\{(\delta_1)_{\lambda=-\lambda_3} + (h_{25} - \lambda_3 h_{22})\{(\delta_2)_{\lambda=-\lambda_3}\} \\
 &\quad + (h_{26} - \lambda_3 h_{23})\{(\delta_3)_{\lambda=-\lambda_3}\} - \beta_2\{(\delta_4)_{\lambda=-\lambda_3}\}]e^{-\lambda_3 x_1}, \\
 R_{24}(x_1) &= [(h_{24} - \lambda_4 h_{21})\{(\delta_1)_{\lambda=-\lambda_4} + (h_{25} - \lambda_4 h_{22})\{(\delta_2)_{\lambda=-\lambda_4}\} \\
 &\quad + (h_{26} - \lambda_4 h_{23})\{(\delta_3)_{\lambda=-\lambda_4}\} - \beta_2\{(\delta_4)_{\lambda=-\lambda_4}\}]e^{-\lambda_4 x_1}, \\
 R_{31}(x_1) &= [(h_{34} - \lambda_1 h_{31})\{(\delta_1)_{\lambda=-\lambda_1} + (h_{35} - \lambda_1 h_{32})\{(\delta_2)_{\lambda=-\lambda_1}\} \\
 &\quad + (h_{36} - \lambda_1 h_{33})\{(\delta_3)_{\lambda=-\lambda_1}\} - \beta_3\{(\delta_4)_{\lambda=-\lambda_1}\}]e^{-\lambda_1 x_1}, \\
 R_{32}(x_1) &= [(h_{34} - \lambda_2 h_{31})\{(\delta_1)_{\lambda=-\lambda_2} + (h_{35} - \lambda_2 h_{32})\{(\delta_2)_{\lambda=-\lambda_2}\} \\
 &\quad + (h_{36} - \lambda_2 h_{33})\{(\delta_3)_{\lambda=-\lambda_2}\} - \beta_3\{(\delta_4)_{\lambda=-\lambda_2}\}]e^{-\lambda_2 x_1}, \\
 R_{33}(x_1) &= [(h_{34} - \lambda_3 h_{31})\{(\delta_1)_{\lambda=-\lambda_3} + (h_{35} - \lambda_3 h_{32})\{(\delta_2)_{\lambda=-\lambda_3}\} \\
 &\quad + (h_{36} - \lambda_3 h_{33})\{(\delta_3)_{\lambda=-\lambda_3}\} - \beta_3\{(\delta_4)_{\lambda=-\lambda_3}\}]e^{-\lambda_3 x_1}, \\
 R_{34}(x_1) &= [(h_{34} - \lambda_4 h_{31})\{(\delta_1)_{\lambda=-\lambda_4} + (h_{35} - \lambda_4 h_{32})\{(\delta_2)_{\lambda=-\lambda_4}\} \\
 &\quad + (h_{36} - \lambda_4 h_{33})\{(\delta_3)_{\lambda=-\lambda_4}\} - \beta_3\{(\delta_4)_{\lambda=-\lambda_4}\}]e^{-\lambda_4 x_1}, \\
 R_{41}(x_1) &= [(h_{44} - \lambda_1 h_{41})\{(\delta_1)_{\lambda=-\lambda_1} + (h_{45} - \lambda_1 h_{42})\{(\delta_2)_{\lambda=-\lambda_1}\} \\
 &\quad + (h_{46} - \lambda_1 h_{43})\{(\delta_3)_{\lambda=-\lambda_1}\}]e^{-\lambda_1 x_1}, \\
 R_{42}(x_1) &= [(h_{44} - \lambda_2 h_{41})\{(\delta_1)_{\lambda=-\lambda_2} + (h_{45} - \lambda_2 h_{42})\{(\delta_2)_{\lambda=-\lambda_2}\} \\
 &\quad + (h_{46} - \lambda_2 h_{43})\{(\delta_3)_{\lambda=-\lambda_2}\}]e^{-\lambda_2 x_1},
 \end{aligned}$$

$$R_{43}(x_1) = [(h_{44} - \lambda_3 h_{41})\{(\delta_1)_{\lambda=-\lambda_3}\} + (h_{45} - \lambda_3 h_{42})\{(\delta_2)_{\lambda=-\lambda_3}\} + (h_{46} - \lambda_3 h_{43})\{(\delta_3)_{\lambda=-\lambda_3}\}]e^{-\lambda_3 x_1},$$

$$R_{44}(x_1) = [(h_{44} - \lambda_4 h_{41})\{(\delta_1)_{\lambda=-\lambda_4}\} + (h_{45} - \lambda_4 h_{42})\{(\delta_2)_{\lambda=-\lambda_4}\} + (h_{46} - \lambda_4 h_{43})\{(\delta_3)_{\lambda=-\lambda_4}\}]e^{-\lambda_4 x_1},$$

$$R_{51}(x_1) = [(h_{54} - \lambda_1 h_{51})\{(\delta_1)_{\lambda=-\lambda_1}\} + (h_{55} - \lambda_1 h_{52})\{(\delta_2)_{\lambda=-\lambda_1}\} + (h_{56} - \lambda_1 h_{53})\{(\delta_3)_{\lambda=-\lambda_1}\}]e^{-\lambda_1 x_1},$$

$$R_{52}(x_1) = [(h_{54} - \lambda_2 h_{51})\{(\delta_1)_{\lambda=-\lambda_2}\} + (h_{55} - \lambda_2 h_{52})\{(\delta_2)_{\lambda=-\lambda_2}\} + (h_{56} - \lambda_2 h_{53})\{(\delta_3)_{\lambda=-\lambda_2}\}]e^{-\lambda_2 x_1},$$

$$R_{53}(x_1) = [(h_{54} - \lambda_3 h_{51})\{(\delta_1)_{\lambda=-\lambda_3}\} + (h_{55} - \lambda_3 h_{52})\{(\delta_2)_{\lambda=-\lambda_3}\} + (h_{56} - \lambda_3 h_{53})\{(\delta_3)_{\lambda=-\lambda_3}\}]e^{-\lambda_3 x_1},$$

$$R_{54}(x_1) = [(h_{54} - \lambda_4 h_{51})\{(\delta_1)_{\lambda=-\lambda_4}\} + (h_{55} - \lambda_4 h_{52})\{(\delta_2)_{\lambda=-\lambda_4}\} + (h_{56} - \lambda_4 h_{53})\{(\delta_3)_{\lambda=-\lambda_4}\}]e^{-\lambda_4 x_1},$$

$$R_{61}(x_1) = [(h_{64} - \lambda_1 h_{61})\{(\delta_1)_{\lambda=-\lambda_1}\} + (h_{65} - \lambda_1 h_{62})\{(\delta_2)_{\lambda=-\lambda_1}\} + (h_{66} - \lambda_1 h_{63})\{(\delta_3)_{\lambda=-\lambda_1}\}]e^{-\lambda_1 x_1},$$

$$R_{62}(x_1) = [(h_{64} - \lambda_2 h_{61})\{(\delta_1)_{\lambda=-\lambda_2}\} + (h_{65} - \lambda_2 h_{62})\{(\delta_2)_{\lambda=-\lambda_2}\} + (h_{66} - \lambda_2 h_{63})\{(\delta_3)_{\lambda=-\lambda_2}\}]e^{-\lambda_2 x_1},$$

$$R_{63}(x_1) = [(h_{64} - \lambda_3 h_{61})\{(\delta_1)_{\lambda=-\lambda_3}\} + (h_{65} - \lambda_3 h_{62})\{(\delta_2)_{\lambda=-\lambda_3}\} + (h_{66} - \lambda_3 h_{63})\{(\delta_3)_{\lambda=-\lambda_3}\}]e^{-\lambda_3 x_1},$$

$$R_{64}(x_1) = [(h_{64} - \lambda_4 h_{61})\{(\delta_1)_{\lambda=-\lambda_4}\} + (h_{65} - \lambda_4 h_{62})\{(\delta_2)_{\lambda=-\lambda_4}\} + (h_{66} - \lambda_4 h_{63})\{(\delta_3)_{\lambda=-\lambda_4}\}]e^{-\lambda_4 x_1},$$

$$R_{71}(0) = [(\gamma - \lambda_1)\{(\delta_4)_{\lambda=-\lambda_1}\}]e^{-\lambda_1}, \quad R_{72}(0) = [(\gamma - \lambda_2)\{(\delta_4)_{\lambda=-\lambda_2}\}]e^{-\lambda_2},$$

$$R_{73}(0) = [(\gamma - \lambda_3)\{(\delta_4)_{\lambda=-\lambda_3}\}]e^{-\lambda_3}, \quad R_{74}(0) = [(\gamma - \lambda_4)\{(\delta_4)_{\lambda=-\lambda_4}\}]e^{-\lambda_4},$$

$$\Delta_1 = \begin{pmatrix} 0 & R_{12}(0) & R_{13}(0) & R_{14}(0) \\ 0 & R_{22}(0) & R_{23}(0) & R_{24}(0) \\ 0 & R_{32}(0) & R_{33}(0) & R_{34}(0) \\ r^* & R_{72}(0) & R_{73}(0) & R_{74}(0) \end{pmatrix}, \quad \Delta_2 = \begin{pmatrix} R_{11}(0) & 0 & R_{13}(0) & R_{14}(0) \\ R_{21}(0) & 0 & R_{23}(0) & R_{24}(0) \\ R_{31}(0) & 0 & R_{33}(0) & R_{34}(0) \\ R_{71}(0) & r^* & R_{73}(0) & R_{74}(0) \end{pmatrix},$$

$$\Delta_3 = \begin{pmatrix} R_{11}(0) & R_{12}(0) & 0 & R_{14}(0) \\ R_{21}(0) & R_{22}(0) & 0 & R_{24}(0) \\ R_{31}(0) & R_{32}(0) & 0 & R_{34}(0) \\ R_{71}(0) & R_{72}(0) & r^* & R_{74}(0) \end{pmatrix}, \quad \Delta_4 = \begin{pmatrix} R_{11}(0) & R_{12}(0) & R_{13}(0) & 0 \\ R_{21}(0) & R_{22}(0) & R_{23}(0) & 0 \\ R_{31}(0) & R_{32}(0) & R_{33}(0) & 0 \\ R_{71}(0) & R_{72}(0) & R_{73}(0) & r^* \end{pmatrix},$$

$$\Delta = \begin{pmatrix} R_{11}(0) & R_{12}(0) & R_{13}(0) & R_{14}(0) \\ R_{21}(0) & R_{22}(0) & R_{23}(0) & R_{24}(0) \\ R_{31}(0) & R_{32}(0) & R_{33}(0) & R_{34}(0) \\ R_{71}(0) & R_{72}(0) & R_{73}(0) & R_{74}(0) \end{pmatrix}.$$