1925. Analytical approaches to vibration analysis of circular, annular and sectorial plates subjected to classical and arbitrary boundary conditions – a literature survey

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Abstract. Plates are one of the most important structural components used in many industries like aerospace, marine and various other engineering fields and thus motivate designers and engineers to study the vibration characteristics of these structures. A lot of research work and studies have been done to study its vibration characteristics. This paper is a review of existing literature on vibration analysis of circular, annular and sector plates. The aim of this paper is to compile prominent studies related to circular, annular and sector plates subjected to classical and arbitrary boundary conditions under different supports and loadings. This review also identifies the analytical methods and approaches used to study the vibration characteristics of circular, annular and sector plates based on classical plate theories, Mindlin plate theory and higher order shear deformation theories. Few important citations related to functionally graded circular, annular and sector plates have also been included. Apart from helping researchers and engineers to identify relevant literature quickly and easily, this review will also help them to apply some of these analytical methods to study the vibration characteristics of other 2D and 3D built up and coupled structures.

Keywords: vibrations, circular plates, annular plates, sector plates, natural frequency, mode shapes, arbitrary boundary conditions.

1. Introduction

Normally the circular, annular and sector plates are treated separately because under different boundary conditions different solutions are obtained for these types of plates. Although a lot of research work and studies have been done to study its vibration characteristics however in terms of publications in the literature far less studies exist comparatively to their rectangular counterparts. The early research on circular, annular and sector plates was based on classical plate theory (CPT). Owing to assumptions in CPT the error in the Eigen frequencies increased with plate thickness thus limiting its application on moderately thick and thick circular, annular and sector plates. Incorporating the effect of shear deformation and rotary inertia various shear deformation theories were proposed from time to time to increase the accuracy of solutions for these plates obtained earlier using CPT. A detailed earlier research on these plates can be found in Leissa's book 1973. Over the past decades a lot of studies have been performed to investigate the vibration characteristics of circular, annular and sector plates. This literature survey is aimed at highlighting up to date key research work done on the vibration characteristics of these plates.

2. Circular plates

Deresiewicz and Mindlin were the first who studied flexural vibrations of axially symmetric circular disks and obtained mode shapes for free circular disks using the classical thin plate theory as well as Mindlin plate theory [1]. Based on same Mindlin plate theory and using Chebyshev collocation technique, Soni et al. studied the free vibration of axisymmetric orthotropic non-uniform circular discs with shear deformation [2]. Azimi employed Receptance method to study the free vibration of elastically restrained circular plates [3]. Later Liew et al. used well known Rayleigh-Ritz method to study the vibration characteristic of circular Mindlin plates under different boundary conditions with multiple internal ring supports. In another similar study he investigated the vibration behavior of these plates under isotropic in-plane pressure [4, 5]. In another important research Y. Xiang et al. developed a general numerical procedure to study the vibration response of thick circular plate systems. They studied the vibration behavior of circular Mindlin plates with internal ring stiffeners. Mindlin first order shear deformation plate theory was employed to cater for the plate shear deformation and rotation whereas Engesser theory was used to take into effect the shear deformation and torsional effects in stiffeners [6].

B. Singh and S. M. Hassan studied the transverse vibration of a circular plate with arbitrary thickness variation. The distinguishing feature of their investigation was that instead of using the linear or quadratic variation of the thickness, they approximated the thickness by measuring the thickness of plate at suitable set of sample points and then used interpolation among those sample points to get the thickness polynomial. Extensive results were reported for clamped, simply supported and completely free boundary conditions [7]. Three dimensional vibration analyses of thick circular plates were done by J. So and A. W. Leissa using the Rayleigh-Ritz method [8]. Trigonometric functions in the circumferential coordinate and algebraic polynomial in the radial and axial coordinate were used as admissible function for the displacement components. All 3-D modes (flexural thickness-shear, in plane stretching and torsional) were analyzed and extensive and accurate frequency results were reported for completely free circular plates with different thickness to diameter ratios and passion ratio. Wu et al. employed differential quadrature method to study the free vibration analysis of solid circular plates with thickness varying in radial direction and elastic boundary conditions. Extensive numerical results were reported for circular plates illustrating the versatility and accuracy of the solution technique [9].

Another important research regarding circular plate was performed by Farag et al. They studied the model characteristics of in-plane vibration of circular plates clamped at outer edge. The mode shapes were presented by trigonometric functions in the circumferential directions and by series summation of Bessel functions in the radial direction [10]. In addition to accurate prediction of natural frequency and mode shapes this research also highlighted the coupling between the different circumferential and radial modes and the response of different vibrational modes at the center of plate. It was shown that the center point of the plate vibrates only for modes with unity circumferential wave number. Unlike conventional 2D thin plate theories Jae Hoon Kang in his research employed Rayleigh-Ritz method to study the three dimensional free vibration analyses of thick circular and annular plates with uniform, linear and quadratic thickness variation along the radial edge. He concluded that as the degree of the polynomials used for the thickness variation is increased, the solution converges to exact values [11]. A similar study was performed by Zhou et al. They employed Chebyshev-Ritz to study the three dimensional analysis of circular and annular plates. They used Chebyshev polynomial in terms of cosine function as admissible function. In their research they concluded that not only lower order but higher order frequency eigen frequencies can also be obtained using sufficient number of terms in Chebyshev polynomial [12].

Chen et al. presented a meshless method for the vibration analysis of clamped circular and rectangular plates using Radial basis function [13]. They employed singular value decomposition technique to obtain the eigen values, eigen frequencies and mode shapes at the same time. In another important study Xiang et al. used domain decomposition technique and presented the exact vibration of Circular Mindlin plates with stepped thickness variation. In their study they studied the effect of plate boundary conditions, plate thickness ratios, step locations ratios and step thickness ratios on the vibration characteristics of stepped circular Mindlin plates [14]. In another important study, Zhou et al. used three-dimensional small-strain, linear and exact elasticity theory to investigate the free vibration of thick circular plates resting on Pasternak foundations. They used Chebyshev polynomials as admissible function of the displacement component in each direction and employed Ritz method to find the eigen values and eigen frequencies. It was shown that for the vibration analysis of plates resting on elastic foundations, the validity of two dimensional classical plate theory and first order shear deformation theory not only depends on the thickness of the plates but also on the stiffness of the elastic foundations [15]. In another similar study on thin plates Li Yongqiang and Li Jian used curve strip Fourier p-element to investigate the free vibration of circular and annular plates. Fixed number of polynomial shape functions and variable number of trigonometric shape functions were used to describe the element's nodal displacements and to provide additional freedom to the edges respectively. After comparing the results for circular and annular thin plates it was found that curve strip Fourier p-element gives much more accurate results then the proposed Fourier p- element and the finite strip element [16].

Chan I1, in his research, developed frequency equation for the in-plane vibration of uniform thickness isotropic circular plate structure using the stress strains displacement relationships and Hamilton principle. Helmholtz decomposition technique was used to uncouple the coupled equations. Using separation of variable method general equation of motion for circular plates was obtained [17]. Wang et al. used direct displacement method to study the free axisymmetric vibrations of transversely isotropic FGM circular plates based on three dimensional theory. It was shown that the exact frequency for free vibrations can be obtained when the variation in the material properties obey the exponential law. It was also concluded numerically that when the material is homogeneous the exact solution agrees with FEM for arbitrary thickness to radius ratio [18]. Using the discrete singular convolution method and regularized Shannon's delta kernel, Civalek et al., in another study, presented a computationally efficient and accurate numerical method to study the free vibration and bending behavior of thick circular plates based on Mindlin plate theory. Accurate results were yielded using DSC method for free vibration of circular plates [19].

In another study related to circular disks, Bashmal et al. used two dimensional linear plane stress elasticity theory to study the in-plane free vibrations of an elastic and isotropic circular disk and presented a generalized solution for the in-plane vibration characteristics of circular annular disks under all classical boundary conditions. Rayleigh-Ritz method was employed to obtain the eigen values and eigen frequencies [20]. Based on two dimensional linear plane stress theory of elasticity, Bashmal et al. studied the in-plane free vibration of elastic isotropic circular annular disks under combination of different boundary conditions at the inner and outer edges. Frequency equations were presented for different combinations of boundary conditions which can be numerically evaluated to obtain the in plane modal characteristics of circular disks for a wide range of boundary conditions and geometric parameters [21]. Using differential quadrature method, Hashemi et al. [22] studied the buckling and free vibration analysis of radially functionally graded circular plates subjected to uniform in-plane compressive loads and resting on Pasternak elastic foundation. Critical buckling load and fundamental frequency parameters of the circular as well as annular plates were computed for different boundary conditions, elastic constants, sector angles and cut out ratios.

Ravari et al. [23] investigated the in-plane vibration analysis of orthotropic circular plates and derived equation for in-plane vibration of circular annular plates for general boundary conditions. Helmholtz decomposition technique was used to uncouple the coupled equations of motion derived using stress strain displacement relationships. Separation of variables method was employed to solve the equations using the boundary conditions. Z. H. Zhou et al. [24] presented an approach based on the conservation principle of mixed energy to study the natural vibrations of thin circular and annular plates. Using the variational principle of mixed energy, a set of Hamiltonian dual equations with derivatives with respect to the radial coordinate on one side of the equations and to the angular coordinate on the other side was obtained. The separation of variable was employed to solve Hamiltonian dual equations of eigen value problem. Kim et al. presented the exact solution for in-plane natural vibrations of circular plates having outer edges restrained by circumferentially distributed radial and tangential stiffness. The mode shapes were expressed by trigonometric functions with a number of nodal diameters in the circumferential direction and mode functions in the radial direction in the said study [25].

Hashemi et al. [26] presented an analytical solution based on exact new closed form solution procedure for free vibration analysis of stepped circular functionally graded plates using Mindlin's first order shear deformation theory. Based on the domain decomposition technique highly coupled governing partial differential governing equations of motion for freely vibrating functionally graded plates were exactly solved by introducing the new potential functions as well as using the method of separation of variables. Results showed that the step parameters (step thickness ratios and step locations) play a significant role in determining the vibration behavior of functionally graded plates especially for higher vibrating modes.

2.1. Circular sector plates

Though most of the research work on circular plates vibration also included analysis on its sector counter parts however few important researches on circular sector plates have also been conducted separately. Cheung et al. studied the vibration analysis of thick layered circular and sector plates and obtained natural frequencies using Finite Element Method [27]. Guruswamy and Yang employed Mindlin plate theory and proposed a sector finite element to study the dynamic response of fully clamped thick sector plates [28]. In another study Cheung et al. studied the vibration analysis of thick and thin sector plates and obtained frequency parameters for fully clamped and simply supported thick sector plates using three dimensional finite strip model [29]. However quite significant difference in results was obtained for fully clamped case after comparing it with the results of Guruswamy and Yang.

M. Es'Haghi [30] employed Reddy's third order shear deformation theory to give exact closed form characteristic equation for thick circular sector plates with hard simply supported along radial edges and other different combinations of classical boundary conditions. Hamiltonian principle was used to derive governing equations for free vibrating sector plates. The effect of thickness radius ratio, inner-outer radius ratio, sector angler and boundary conditions on the natural frequencies of circular plates was discussed.

X. Shi et al. [31] presented a unique unified method for free vibration analysis of circular annular and sector plates with arbitrary boundary conditions. The unification of different value problems (circular, annular and sector plates) was accomplished by applying a set of coupling springs to ensure appropriate continuity equations along the radial edges. This unified method can easily be applied to sector plates with an arbitrary inclusion angle of 2π.

3. Annular plates

In case of annular plates, valuable results were obtained by Irie et al. [32] after analyzing the vibration response of annular Mindlin plate using Bessel function in radial direction and trigonometric function in tangential direction as admissible function for displacement field. Using Chebyshev collocation technique, Gupta et al. studied the axisymmetric vibration response of polar orthotropic annular Mindlin plates with non-uniform thickness [33]. Dauge et al. compared the eigen frequencies of three dimensional thin plates and eigen frequencies of two dimensional Reissner-Mindlin plate models mathematically and analytically. In mathematical analysis they defined an asymptotic expansion for the eigen frequencies in power series of the thickness parameter and obtained new results for orthotropic Reissner-Mindlin plate model. From numerical experiments for clamped plates they showed that for isotropic thin plates the Reissner-Mindlin eigen frequencies provide a good approximation to the thin plate frequencies [34].

Using NASTRAN, Cheng et al. [35], investigated the effects of eccentricity, hole size and boundary condition on vibration modes of annular like plates systematically through both global and local analysis. He reported that the existence of eccentricity of the central hole in an annular plate significantly effects the natural frequencies and mode shapes of the structure. This study is very helpful for modal analysis, vibration measurement and damage detection of plate like structures. Using the combination of the vibration theory and transfer matrix method, Liang et al.

[36] investigated the vibration characteristics for a conical shell with an annular or round end plate. The governing equation for this type of structure was expressed in terms of a matrix differential equation. The application of transfer matrix method on such structures was verified by obtaining accurate natural frequencies and mode shapes for conical shell with annular end plate.

Based on small strain and linear elasticity theory an important three dimensional free vibration analysis of annular plates resting on elastic foundation was done by Hashemi et al. [37]. They used polynomials-Ritz approach to study the annular plates under different combinations of clamped, free and simply supported boundary conditions at the inner and outer edges. They studied the effect of foundation stiffness, thickness-radius ratio, cutout ratio and various boundary conditions on the ill-conditioning of mass matrix and on the vibrational response of the annular plates. Houmat in another study investigated the in-plane vibration analysis of plates with arbitrary curvilinear plan-forms by a trigonometrically enriched curved triangular p element. He calculated frequency values for closed and open sectorial plates and compared it with exact solution. He found out that the method converges very quickly to exact solution as the order of trigonometric function is increased and show very high accuracy then the linear and isoparametric quadratic triangular finite elements solution with less degree of freedom [38].

Civalek et al. investigated the free vibration of annular Mindlin plates with free inner edge using discrete singular convolution method [39]. Later in another study Hashemi et al. gave an exact closed form frequency equation for free vibration analysis of annular moderately thick functionally graded plates based on Mindlin's first order shear deformation plate theory. Circular plates were also investigated using the same method. They employed Hamilton principle to derive the equilibrium equations governing the dynamic stability of plate and its natural boundary conditions [40]. Based on Reddy third order shear deformation theory Hashemi et al. provided an exact analytical solution for free vibrations of thick circular annular plates whose upper and lower surfaces are in contact with piezoelectric layer. Natural frequencies were calculated for classical boundary conditions at the inner and outer edges of the plate by solving the electromechanical governing equations of motion [41].

Bashmal et al. used Rayleigh-Ritz method to study the in-plane free vibration analysis of an annular disk with point elastic support. It was concluded from the results that addition of point clamped support causes some of the higher modes to split into two different frequencies with different mode shapes [42]. A new method for the free vibration analysis of isotropic rectangular and annular Mindlin plates with damaged boundary conditions was presented by Sari et al. [43]. They used Chebyshev collocation method to obtain the natural frequencies of Mindlin plates with damaged clamped boundary conditions. The analysis technique presented by the authors lead to an efficient technique for health monitoring of structures in which joint or boundary damage can play a significant role in dynamic characteristics. Bisadi et al. [44] presented closed form solution for free vibration of thick annular plates based on Reddy's higher order shear deformation theory. Hamiltonian and principle of minimum potential energy were applied to derive the equations of dynamic equilibrium and natural boundary conditions of the plate. Benchmark results were established for the transverse displacement of annular thick plates.

In another related study Yas and Tahouneh performed 3-D free vibration analysis of functionally graded thick annular plates resting on elastic foundation using differential quadrature method. New benchmark results were established which includes the effects of elastic coefficients of foundation, boundary conditions, material and geometric parameters [45].

Eftekhari et al. [46] gave an efficient and accurate variational formulation to study the vibration phenomena of non-uniform thickness thin and thick plates with edges elastically restrained against translation and rotation. Ritz method was employed to reduce the governing partial differential equation to ordinary differential equation. Accurate solutions were achieved using few Ritz terms for all the cases considered. Based on classical plate theory assumptions, Bhaskara Rao et al. [47] gave an exact solution of free lateral vibrations of annular plates with inner and outer edges elastically restrained and resting on Winkler type elastic foundation. The effect of varying values of rotational and translational spring stiffnesses, foundation restraint and the radius of the annular plate on vibration characteristics of annular plates was studied in detail.

Xianjie Shi et al. [48] proposed a generalized Fourier series method to study the in-plane vibrations of annular plates with arbitrary boundary conditions along each of its edges. Regardless of the boundary conditions, the in-plane displacement fields were invariantly expressed as a new form of trigonometric series expansions with a drastically improved convergence as compared with the conventional Fourier series. All the unknown expansion coefficients were treated as the generalized coordinates and determined using the Rayleigh-Ritz technique. Unlike most of the other existing methods, this unique generalized Fourier series method can be readily and universally applied to a wide spectrum of in-plane vibration problems involving different boundary conditions, varying material, and geometric properties with no need of modifying the basic functions or adapting solution procedures

M. S. Sari employed Eringen's nonlocal elasticity theory to study the free vibration of Mindlin annular plates at micro/nano scale [49]. He used nonlocal differential constitutive relations of Eringen to derive the governing equations which were further solved by Chebyshev collocation method to get the eigen frequencies. Apart from the effect of cutout ratio, thickness to radius ratio and boundary conditions, the nonlocal effect on the natural frequency of mindlin plates was studied in this research.

3.1. Annular sector plates

In another similar study Sirinivasan et al. [50] employed Integral equation technique to study the free vibration of annular sector Mindlin plate. Misuzawa et al. used finite strip method and finite prism method based on Mindlin plate theory and three dimensional elastic plate theory respectively for studying the vibration characteristic of thick annular sector plates [51]. Xiang et al. employed pb-2 Rayleigh-Ritz method to analyze vibration characteristics of Mindlin annular sector plates. Valuable results with different boundary conditions, cutout ratios, sector angles and relative thickness ratios were obtained. Furthermore, they also investigated the vibration characteristics of Mindlin annular sector plates with internal radial line and circumferential arc supports [52, 53]. Later in another study Liew and Yang examined the annular plates by using linear small-strain three dimensional elasticity theory. A polynomial-Ritz model to approximate the displacement in cylindrical coordinates was presented. The effect on vibration characteristic due to change in boundary conditions and thickness was investigated. Analytical results were compared with the results obtained from FEM software NASTRAN and the frequency parameters and three dimensional mode shapes were presented graphically [54].

In another study a hybrid numerical approach was proposed by S. Kitipornchai et al. in which a combination of the Rayleigh-Ritz method and the Lagrange multiplier method was used to analyze the flexural vibrations of corner supported Mindlin plates of arbitrary shape. They used pb-2 shape functions and to cater for arbitrary shapes and Lagrange multiplier to apply the zero lateral deformation constraint at plate corner. This method is applicable to arbitrary shaped plates with corner supports [55]. McGee et al. [56] derived the first known exact solution for the Mindlin annular sector plates simply supported at radial edges and general boundary conditions along the circular edge. They used ordinary and modified Bessel functions of the first and second kind to obtain the general solution of differential equation of motion. Employing the spline strip method, Misuzawa et al. [57] analyzed the vibration of annular sector Mindlin continuous plates with intermediate arc supports. Stable convergence and good accuracy in results was obtained using the high order spline strip models.

Liu et al. employed differential quadrature method to study the free vibration of moderately thick sector plates based on Mindlin's first order shear deformation theory. They discussed the effect of sector angle and the relative thickness ratio on the frequency parameters in their study [58]. In another similar study Liew and Liu used same differential quadrature method to study the vibration characteristics of shear deformable annular sector plates. The first six natural frequencies were calculated for the plates with arbitrary combination of free, clamped and simply supported boundary conditions with different relative thickness, sector angle and cutout ratios [59].

Wang et al. in his research [60], on sectorial plates simply supported at radial edges and simply supported, clamped or free circular edges, gave exact relationships between the bending solutions of sectorial plates based on classical thin plate theory and Mindlin plate theory. This relationship allow easy conversion of the thin plate solution to corresponding thick plate solution without solving the more complicated bending equations for Mindlin plates. Later Wong et al. studied the vibration analysis of annular sector plates by employing mode subtraction method. They studied the changes in displacement mode shapes with respect to different hole sizes under different boundary conditions. Experimentally validated numerical results were reported in the current study [61]. In another important study, Houmat [62] formulated a sector Fourier p-element for sector plates using fixed number of cubic polynomial shape functions and variable number of trigonometric hierarchical shape functions to describe the degree of freedom of the nodes and to give additional freedom to the edges respectively. He studied the free vibrations of sectorial plates using this element and showed that the solution converge very quickly to the exact values with increase in trigonometric terms and very accurate values of frequency parameters are obtained.

Houmat [63], presented an annular sector solid hierarchical finite element to study the three dimensional free vibration analyses of annular sector plates. He used fixed number of linear polynomial shape functions and variable higher order shape functions to express the element nodal displacement and to provide additional freedom to the edges, face sand interior of the element respectively. Huang et al. [64], studied the effect of modulus of elasticity and shear modulus on the vibration characteristics of sector plates with free and clamped boundary conditions at the circumferential edge. They employed Frobenius method and presented an exact analytical solution for polarly orthotropic Mindlin sector plates simply supported at radial edges. The exact solution not only satisfied the boundary condition but also the regularity conditions at the vertex of the radial edges.

Seok et al. used a variational approximation method to study the in plane and out of plane vibration characteristics of annular sector cantilever plates [65, 66]. This variational method yielded very accurate results when compared with FEM and also it provides help in understanding in terms of waves that makeup the vibration which is never provided by any other method. Isaac Elishakoff and Dave Chandra studied the vibration tailoring of heterogeneous beams and annular plates. In their study they highlighted that a heterogeneous Euler-Bernoulli beams fixed at both ends and Kirchoff-Love annular plates that is fixed along inner and outer radius possess a common fundamental mode shapes which is a fourth order polynomial [67].

Aghdam et al. [68], employed Extended Kantorovich Method (EKM) to obtain accurate approximate closed form solution for bending of thin sector plates with fixed edges subjected to uniform and non-uniform loading. Using EKM the fourth order partial differential governing equation was converted to two separate ordinary differential equations in terms of radial and circumferential components which were further solved to obtain the exact closed form solution.

Based on Mindlin plate theory, an exact analytical solution for free vibrations of a transversely isotropic sector plate simply supported at radial edges and arbitrary conditions along the circular edges was presented by Jomehzadeh et al. Boundary layer function was employed to uncouple the three coupled governing equations which were then solved to obtain the desired eigen values and eigen frequencies. The effect of boundary conditions, variation in sector angle and thickness-radius ratios on the frequency parameters have been discussed in detail in this research [69]. In another similar study Jomehzadeh et al. employed Mindlin first order shear deformation theory and presented an analytical solution for free vibrations of transversely isotropic moderately thick annular sector plates simply supported at radial edges and arbitrary conditions along the circular edge. By introducing boundary layer function in their solution, the three coupled equations of motion were reduced to two uncoupled equations containing integer and non-integer order Bessel and/or modified Bessel functions of the first and second kind [70].

Based on three dimensional small strain linear elasticity theory, D. Zhou et al. [71] employed Chebyshev-Ritz method to study the three dimensional free vibration of annular sector plates with various boundary conditions. Chebyshev polynomials satisfying the necessary boundary conditions were used as admissible functions. This research provides a full vibration spectrum for the thick annular sector plates which cannot be given by the classical plate theory or two dimensional theories. Based on classical Kirchoff plate theory Gurses et al. [72] employed discrete singular convolution method and Regularized Shannon Delta (RSD) kernel to study the vibration characteristics of sector plates. The effect of sector angle, boundary conditions and mode numbers on the frequency parameters was investigated.

In another similar study Hashemi et al. employed differential quadrature method to get natural frequencies of simply supported and clamped circular/annular sectorial functionally graded plates having non uniform thickness and resting on elastic foundation [73]. In another study on FGM plates Baferani et al. [74] introduced an exact analytical method for free vibration analysis of functionally graded annular sector plates simply supported at radial edges, arbitrary boundary conditions along the circular edges and resting on elastic foundations. In their studies accurate non dimensional frequency is presented for over a wide range of sector angles, aspect ratios and different powers of FG materials. Other similar studies were performed by Mirtalaie et al. They used differential quadrature method to study the free vibrations of functionally graded thin and thick annular sector plates and also identified the effect of boundary conditions, volume fraction exponent and variation of Poisson's ratio on the free vibration behavior of FGM plates [75, 76]. Hejripour and Saidi also used differential quadrature method to study the non- linear free vibration of isotropic annular sector plates. They used harmonic balance method to obtain the eigen value system which was further solved by iterations. It was concluded that the simply supported plate is more sensitive to geometric parameters then the clamped one [77].

Using Kantorovich method, Fereidoon et al. investigated bending of functionally graded annular sector plates. Poisson ratio of the plate was assumed to be constant however the modulus of elasticity varied in the radial direction according to the power law function. Fast convergence was observed in the solution with application of Kantorovich method and classical thin plate theory [78]. Shi et al. [79] presented an analytical method to study the dynamic analysis of annular sector plates with arbitrary elastic boundary supports. Arbitrary boundary supports were achieved by using two sets of boundary restraining springs. All the classical boundary conditions (free and clamped) could easily be achieved by setting the stiffnesses of the restraining springs either equal to zero or infinity respectively. In their research a double Fourier series solution was obtained for dynamic analysis of the annular plate structure using the Rayleigh-Ritz method.

In another important study Xianjie Shi et al. developed an analytical Fourier series method to investigate the three dimensional vibration analyses of annular sector plates with arbitrary boundary conditions. The displacement functions were invariably expressed as improved trigonometric series which converges uniformly and polynomially over the entire solution domain. Rayleigh-Ritz method was employed to solve the unknown expansion coefficients in the displacement functions [80].

Bahrami et al. [81] investigated the free vibrations of circular, annular and sector membranes using the wave propagation approach. The propagation and reflection matrices for circular, annular and sectorial membranes were derived and then combined to provide a systematic approach to obtain the natural frequencies. This solution method can be extendible to plates also. In another study S. Bahrami et al. [82] developed a Spectral Split Method (SSM) in frequency domain to analyze wave propagation in annular Levy-type plates with simple boundary conditions on radial edges and arbitrary boundary conditions on circular edges with constant and variable thickness. The dynamic differential equation of annular sector plate was further solved using Bessel's equations, modified Bessel's equation and separation of variable technique.

Liang et al. [83] proposed a semi analytical method composed of state space method; differential quadrature method and the numerical inversion method of Laplace transform to study the transient response of functionally graded annular sector plates with arbitrary circular boundary conditions. In another study on functionally graded annular sector plates Zafarmand et al. [84] investigated the three dimensional static and dynamic analysis of two directional functionally graded thick sector plates. The governing equations based on three dimensional theory of elasticity were solved using 3D graded finite element method based on Hamilton principle and Rayleigh-Ritz energy method. For static case uniform pressure loading was taken into account whereas for the dynamic case impact loading was considered and the effect of material gradient index, boundary conditions and thickness to radius ratio on the static and dynamic response of annular sector plate was discussed in detail.

4. Conclusion

This literature survey aims at compiling of prominent research work done on vibration analysis of circular, annular and sector plates subjected to various combinations of classical and general boundary conditions. Various analytical methods and approaches have been identified and discussed in this paper to help future researchers to extend and apply these methods on other 2D and 3D complex built up and coupled structures. This compilation will also help designers and engineers to get access to prominent work done in the field of circular, annular and sector plates quickly and easily.

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