An Analysis of Gear Based on Isogeometric Analysis

Qingsi Cheng[1]**, Guolai Yang**[2] **and Jia Lu**[3]
[1, 2] School of Mechanics, Nanjing University of Science and Technology, Nanjing, 210094, China.
[3] Department of Mechanical & Industrial Engineering, The University of Iowa, Iowa, 52242, USA.

E-mail: qingsicheng@gmail.com

Abstract. Although the CAGD has shown its ability to represent the geometry at a very high precision, the error is inevitable in FEA due to its imprecise mesh. This paper analyzes the stress state of a gear using NURBS-based IGA (Isogeometric Analysis). And the results show great superiority of IGA on efficiency and accuracy compared with traditional FEM (Finite Element Method). The contours of involute gear is non-rational curve and the general way to represent the geometry is the tracing point method. It is almost impossible to accurately descript the involute at the mathematical level either, however, NURBS can supply exact representation of the involute within acceptable error range. As a result, the NURBS-based IGA offers an effective and accurate calculation for gear analysis. Unlike the FEM, the IGA conducts mechanical analysis directly on NURBS geometry and it skips the step of meshing which will reduce a lot of workload.

1. Introduction

Gears are so widely used that they spread almost every corner of our life as transmission components. In the process of gear design, failure analysis and strength verification are necessary to ensure the safety, and the reliable analysis result depends on the accuracy of stress simulation in actual work conditions. It's no longer the problem to exactly represent the geometry in CAGD, but the error comes once the geometry is replaced by the mesh especially for smooth and curved geometry. In traditional finite element analysis, a large number of elements is often required to adequately capture the geometry [1]. In fact for some common shapes such as circles, an exact description can only be achieved in limit of mesh refinement. The contradiction between the number of elements and the amount of calculation is one of the fundamental disadvantages.

In this work, we present a different method of gear analysis, the Isogeometric Analysis Method wherein a uniform geometric representation is utilized for both geometric description and analysis. We use NURBS here which can provide a globally smooth and yet locally supported basis functions to support both accurate and convenient geometric description. The IGA can directly utilize the geometric models from CAD without resorting to another layer of geometric approximations [2-5]. This is critically beneficial to modern computer aided engineering.

The article begins with an introduction to B-spline and NURBS which are the foundation of IGA. We fit the involute curve by using NURBS interpolation and set up the gear model. The IGA method is applied to analyse the stress of dedendum and the results are compared with traditional FEA (ABAQUS).

2. B-Splines and NURBS

2.1. B-spline

B-splines are constructed by taking a linear combination of B-spline basis functions, just as in traditional FEA. The basis functions can be constructed from a knot vector written as:

$$\mathbf{U} = \left\{ \mathbf{u}_1 = 0; \ldots; \mathbf{u}_i; \ldots; \mathbf{u}_{n+p+1} = 1 \right\}^{\mathrm{T}} \tag{1}$$

The knot vector **U** is a non-decreasing sequences of coordinates in the parametric space, and it can

[2] Corresponding author

be considered as open if its first and last knot are repeated p+1 times. With the given knot vector **U**, the B-splines basis functions $N_{i,p}$ are defined by the following recursive formula:

$$\begin{cases} N_{i,p} = \begin{cases} 1, & u_i \le u < u_{i+1} \\ 0, & \text{otherwise} \end{cases} \quad \text{for } p = 0 \\ N_{i,p}(u) = \dfrac{u - u_i}{u_i - u_i} N_{i,p-1}(u) + \dfrac{u_{i+p+1} - u}{u_{i+p+1} - u_{i+1}} N_{i+1,p-1}(u) \quad \text{for } p \ge 1 \end{cases} \tag{2}$$

The formula may encounter the quotient 0/0, in this case, it is assigned to zero.

A cubic example is presented in Figure 1 using the follow knot vector:

$$\mathbf{U} = \{u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8, u_9, u_{10}, u_{11}\}^{\mathrm{T}} = \{0,0,0,0,2,4,6,8,8,8,8\}^{\mathrm{T}} \tag{3}$$

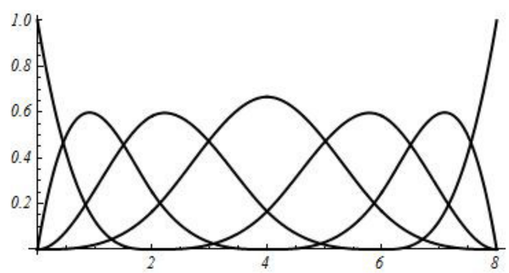


Figure 1. Cubic B-spline basis functions.

More than one knot can be placed at a parametric coordinate. For a p-degree B-spline, an interior knot can be repeated at most p times, and the boundary knots can be repeated at most p+1 times. A knot vector for which the two boundary knots are repeated p+1 times is said to be open. In this case, the basis functions are interpolatory at the first and the last knot [6-8].

With the given n basis functions $N_{i,p}$, $i = 1, 2, \ldots, n$ and corresponding control points $P_i \in \mathbf{R}^d$, $i = 1, 2, \ldots, n$, a piecewise-polynomial B-spline curve can be defined by:

$$C(u) = \sum_{i=1}^{n} N_{i,p}(u) P_i \tag{4}$$

The example shown in Figure 2 is a general p-degree B-spline.

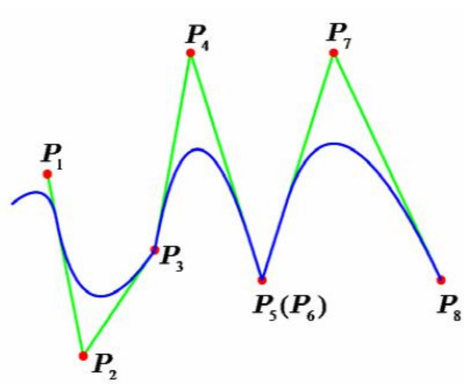


Figure 2. p-degree B-spline.

The knot number m (counting multiplicity) is related to the number of control points n (the same as the number of basis functions since they are in one to one correspondence) and the degree p via the relation:

$$m = n + p + 1 \tag{5}$$

Note that the index i in P_i serves to identify the control point and is not a reference to one of its d components. Piecewise linear interpolation of the control points constitute the so-called control polygon.

2.2. NURBS (Non Uniform Rational B-spline)

From the intuitive point of view, if every control points P_i is set to a weight w_i, A p-th degree NURBS curve is defined by:

$$C(u) = \frac{\sum_{i=1}^{n} N_{i,p}(u) w_i P_i}{\sum_{i=1}^{n} N_{i,p}(u) w_i} \tag{6}$$

From the above formula, the NURBS basis function can be defined as:

$$R_{i,p}(u) = \frac{N_{i,p}(u) w_i}{\sum_{i=1}^{n} N_{i,p}(u) w_i} \tag{7}$$

By using the NURBS basis function, a curve $C(u)$ can be constructed as:

$$C(u) = \sum_{i=1}^{n} R_{i,p}(u) P_i \tag{8}$$

Rational surfaces are defined analogously in terms of the rational basis functions. Taking the tensor product of two one dimensional basis functions, a NURBS surface is represented by:

$$S(u_1, u_2) = \frac{\sum_{i=1}^{n} \sum_{j=1}^{m} N_{i,p}(u_1) N_{j,q}(u_2) w_{i,j} P_{i,j}}{\sum_{i=1}^{n} \sum_{j=1}^{m} N_{i,p}(u_1) N_{j,q}(u_2) w_{i,j}} \tag{9}$$

3. Numerical example

3.1. Geometric modeling

As the contours of involute gear is non-rational curve, the common way to describe the involute is the tracing point method.

With the given set of points on the involute $\{Q_k\}$, $k = 1, 2, \dots, n$, as shown in Figure 3 (a) (The precision of the curve depends on the number of points), we can use a 3-degree NURBS curve (p=3) to interpolate these points. If every point is assigned a parameter value $\bar{u}_k$, the linear system of equations with a $n \times n$ coefficient matrix are defined by:

$$Q_k = C(\bar{u}_k) = \sum_{i=1}^{n} N_{i,p}(\bar{u}_k) P_i \tag{10}$$

If the parameter value $\bar{u}_k$ and the knot vector $\mathbf{U} = \{u_1, u_2, \dots, u_m\}$ have been chosen, we can obtain the coefficient matrix and solve the equations.

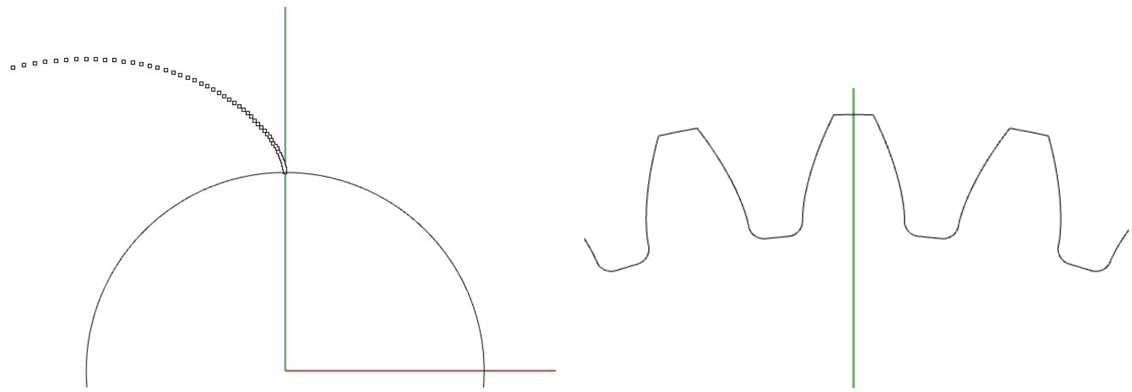


Figure 3. (a): Points on the involute, (b): Geometry model of the gear.

To get the parameter value $\bar{u}_k$, the total chord length d is defined as:

$$d = \sum_{k=1}^{n} \left| Q_k - Q_{k-1} \right| \tag{11}$$

For k=1 and k=n: $\bar{u}_1 = 0,\ \bar{u}_n = 1$

Otherwise:

$$\bar{u}_k = \bar{u}_{k-1} + \frac{\left| Q_k - Q_{k-1} \right|}{d}, k = 2,3,\ldots n-1 \tag{12}$$

This is by far the most common method for its little computation and it works well in most situations.

To get the knot vector **U**, we use the averaging value as following:

$$u_1 = \ldots u_{p+1} = 0,\ u_{m-p} = \ldots = u_m = 1 \tag{13}$$

$$u_{j+p} = \frac{1}{p} \sum_{i=j}^{j+p-1} \bar{u}_i, j = 2,3,\ldots,n-p \tag{14}$$

Thus, the NURBS basis functions $N_{i,p},\ i = 1,2,\ldots,n$ can be calculated, and the linear system of equations are as follow:

$$\begin{bmatrix} 1 & 0 & \ldots & 0 & 0 \\ N_{1,3}(\bar{u}_2) & N_{2,3}(\bar{u}_2) & \ldots & N_{n-1,3}(\bar{u}_2) & N_{n,3}(\bar{u}_2) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ N_{1,3}(\bar{u}_{n-1}) & N_{2,3}(\bar{u}_{n-1}) & \ldots & N_{n-1,3}(\bar{u}_{n-1}) & N_{n,3}(\bar{u}_{n-1}) \\ 0 & 0 & \ldots & 0 & 1 \end{bmatrix} - \begin{bmatrix} P_1 \\ P_2 \\ \vdots \\ P_{n-1} \\ P_n \end{bmatrix} = \begin{bmatrix} Q_1 \\ Q_2 \\ \vdots \\ Q_{n-1} \\ Q_n \end{bmatrix} \tag{15}$$

If we get the control points and set all the weight $w_i = 1, i = 1, 2, \ldots, n$, we can get the NURBS fitting curve. The geometry model of the gear (M4×32) is shown in Figure 3 (b).

3.2. Simulation and results

When the pair of gears start to mesh, the endpoint of the addendum will stand a concentrated force P as shown in Figure 4.

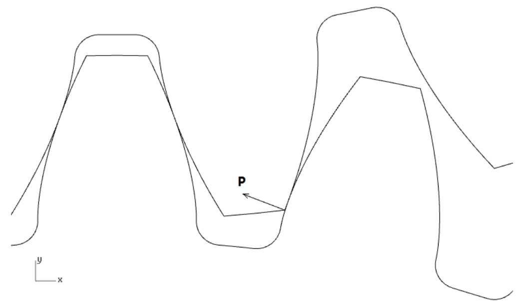
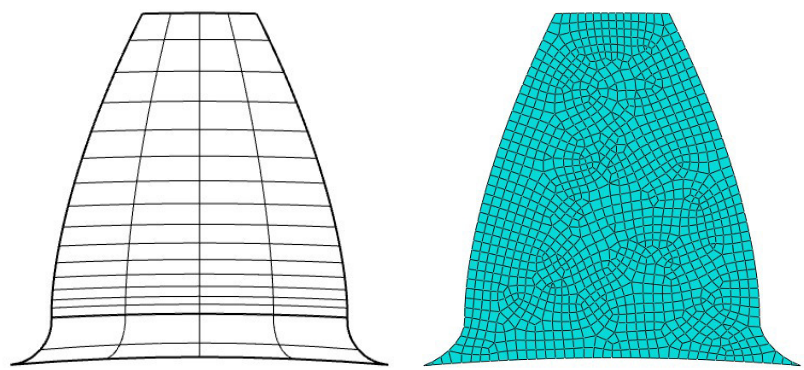


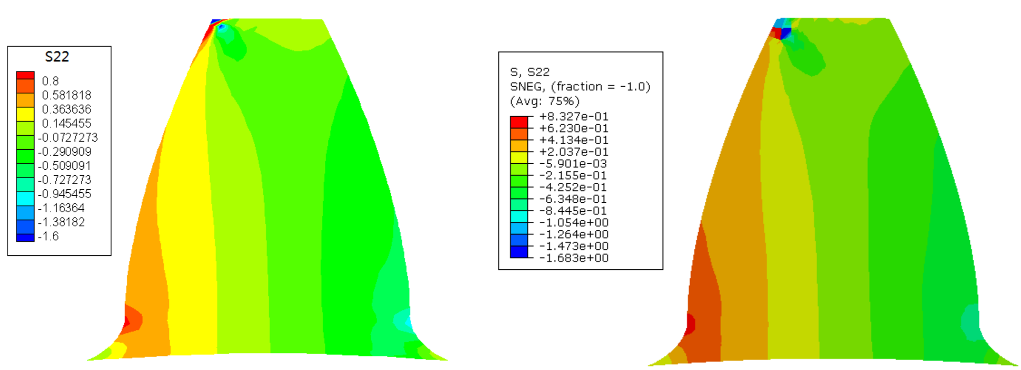
Figure 4. The concentrated force.

For the fatigue strength of gear, the compact load *P* is the greatest factor and the stress of dedendum is the biggest. So we can select one tooth as the analysis model, totally fix the lower boundary and place a concentrated force (*P*=1N) at the endpoint of the addendum. Figure 5 shows the NURBS mesh and the FEA model. Figure 6 shows the stress contour of IGA and FEA.



(a) (b)

Figure 5. (a): NURBS mesh, (b): FEA model.



(a) (b)

Figure 6. S22 (a): IGA model, (b): FEA model.

We can take different mesh density, collect the maximum stress value of dedendum and compare the computational efficiency and accuracy as shown in Figure 7. It is evident to note that the accuracy of IGA is quite higher then FEA under coarse mesh. The curve smoothness and the efficiency is much better then FEA as the increase of the element number.

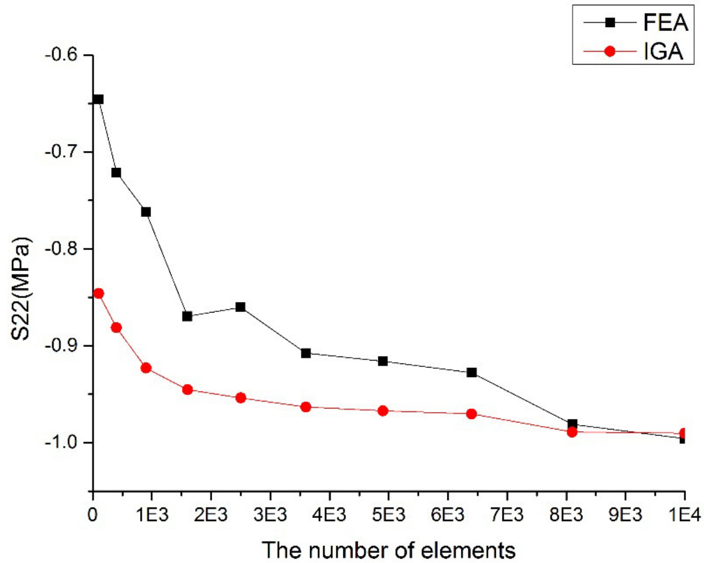


Figure 7. Maximum stress value.

4. Conclusion

In this paper, the IGA method is introduced to analyse the stress state of a gear. The superiority of the IGA is shown by comparing with traditional FEA (simulated in ABAQUS). Future work will focus on the contact analysis of gear, as the NURSB can describe the boundary smoothly which is great beneficial to contact problems between irregular boundaries.

Reference

[1] Lu J 2009 Circular element: Isogeometric elements of smooth boundary [J] *Computer methods in applied mechanics and engineering* **198(30)** 2391–402

[2] T J R Hughes, J A Cottrell and Y Bazilevs 2005 Isogeometric analysis: CAD, finite elements, NURBS, exact geometry and mesh refinement *Comput. Methods Appl. Mech. Engrg.* **194** 4135–95

[3] Y Bazilevs, L Beirão da Veiga, J A Cottrell, T J R Hughes and G Sangalli 2006 Isogeometric analysis: approximation, stability and error estimates for h-refined meshes *Math. Mod. Methods Appl. Sci.* **12** 1031–90

[4] X Zhou and J Lu 2005 NURBS-based Galerkin method and application to skeletal muscle modeling *Proceedings of the 2005 ACM Symposium on Solid and Physical Modeling, SPM'05* ACM Press New York 71–8

[5] J A Cottrell, A. Reali, Y Bazilevs and T J R Hughes 2006 Isogeometric analysis of structural vibrations *Comput. Meth. Appl. Mech. Engrg.* **195** 5257–96

[6] T J R Hughes, A Reali and G Sangalli 2010 Efficient quadrature for NURBS-based isogeometric analysis *Comput. Meth. Appl. Mech. Engrg.* **199** 301–13

[7] L A Piegl and W Tiller 1995 The NURBS Book Springer Berlin

[8] T Elguedj, Y Bazilevs, V M Calo and T J R Hughes 2008 B and F projection methods for nearly incompressible linear and non-linear elasticity and plasticity using higher-order NURBS elements *Comput. Meth. Appl. Mech. Engrg.* **197** 2732–62