

892. An analytical approach to study the dynamic characteristic of beams carrying any type of attachments with arbitrary distributions under elastic constraint boundary supports

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Abstract. An analytical method to study the dynamic characteristic of free vibration of beams carrying any type of attachments with arbitrary distributions on elastic constraint boundary supports is developed in the paper. To obtain an exact solution of the governing function, the displacement function is expressed as a modified Fourier series based on the Euler-Bernoulli beam differential equation, which consists of a standard Fourier cosine series plus several supplementary series used to improve uniform convergence of the series representation. Compared with other techniques, the current method offers a unified solution to entire situations of beams carrying various types of attachments, regarding different distributions and arbitrary boundary conditions. The results of different numerical examples are compared with the results of the references to illustrate the excellent accuracy of the current solution and validate the methodology. Furthermore, the proposed analytical method can be directly extended to calculate the natural frequencies of beam on Pasternak soil and with distribution attachment varying with its length which is never studied before.

Keywords: free vibration, Euler-Bernoulli beam, distributed attachment, elastic constraint boundary.

Nomenclature

$W(x)$	flexural displacement	E	Young's modulus
		I	moment of inertia
L	length of beam	ω	angular frequency
s	cross section	$Uk_i(x), UK_i(x), Um_i(x), Uc_i(x)$	function of location of attachments varying with length of beam
p	mass per unit length		
k_b	linear spring		
K_b	rotational spring		
k_z	stiffness in spring-mass system	N_k, N_K, N_m, N_c	total number of the linear springs, rotational springs, masses and spring-mass systems, seperately
m_z	mass in spring-mass system		
k_0, K_0	linear spring and rotation spring at $x = 0$		
k_L, K_L	linear spring and rotation spring at $x = L$	c^l	coefficients of displacement function
A_n	Fourier series coefficients	$\xi^l(x)$	a set of closed-form sufficiently smooth functions
K_z	stiffness matix		
M_z	mass matrix	λ_n	$n\pi/L$
N	Fourier truncation number	l	index in the special functions and single Fourier series (=1, 2, 3, 4)
i	i th attachment in order		
$\delta(x)$	Dirac delta function		
\bar{k}_0	Winkler modulus of the subgrade reaction	β_i	natural frequency coefficients
Ω_0, Ω_1	non-dimensional soil parameter coefficients		
m_d, k_d	dimensionless parameters of $m_{z,i}$ and $k_{z,i}$	\bar{k}_1	second foundation parameter
k_e	equivalent stiffness of spring-mass system	m_a, k_b	mass parameter and stiffness parameter of the cantilever beam

1 Introduction

A flexural beam carrying one or several elastically mounted attachments, such as engines, motors, oscillators or vibration absorbers, is often encountered in the fields of mechanical, civil and aeronautical engineering. In order to avoid dangerous resonance situations in the design, extensive studies have been carried out with regard to free vibration analysis of beams carrying masses, stiffness, spring-mass system and additional attachments at arbitrary positions with variable boundary conditions.

Low [1] studied natural frequency of an Euler-Bernoulli beam carrying a concentrated mass at an arbitrary location, the results of the fixed-fixed cases are compared with those obtained by experimental method. Chang [2] carried on a research on free vibration of a simply supported beam carrying a rigid mass at middle, obtaining a general solution including both the rotatory inertia of the beam and of the concentrated mass by applying elementary beam theory. Turhan [3] used Rayleigh method to analyze the fundamental frequency of beams carrying a point mass. Gürgöze [4] presented two alternative formulations for the frequency equation of a cantilever Euler-Bernoulli beam with several spring-mass systems. The finite element method is used to study the dynamic behaviors of a beam carrying a number of spring-mass systems with mass of each helical spring considered by Wu & Hsu [5]. Lin & Tsai [6] determined natural frequencies and mode shapes of Euler-Bernoulli multi-span beam carrying multiple spring-mass systems. Zhou & Ji [7] studied dynamic characteristics of a beam with continuously distributed spring-mass, dividing the coupled system into several segments and considering the distributed spring-mass and the beam in each segment being uniform. Furthermore, the influences of variable attachments on slender beam vibrations are studied in [8–16].

The target of all of the researches mentioned above is to obtain the solution of natural frequencies of beam carrying different types of attachments with various distributions by using different computing methods. But the technique, which can entirely deal with the natural frequencies problem of beams attached by any type of distributed attachments with arbitrary boundary condition, is rarely studied to the best of the authors' knowledge. Two articles [17-18] which give details of approach to analyze free vibration of beam with attachments universally are merely found, but the situations of beams with distributed attachments are not included.

In the present manuscript, an analytic method is proposed to give an exact solution of the free vibration of Bernoulli-Euler beam with arbitrary boundary conditions, carrying a finite number of any types of attachments at arbitrary positions. The generality of this approach is based on research of Li [19-20] who studied free vibration of beam and plate of arbitrary boundary condition, using translational and rotational springs at both ends, which allow us to represent all the possible combinations of classical boundary conditions, as well as elastic restraints. In order to validate the methodology, some results of numerical examples are compared with the solutions obtained in references. Other numerical results are presented to show the influence of the various distribution types on the natural frequencies of the combined system.

2 Theory

The structure of the mechanical system to be studied is shown in Fig. 1. The system consists of stiffness supported Bernoulli-Euler beam of bending rigidity EI , length L , area of cross section s and mass per unit length ρ , carrying distributed attachments. The attachments include linear springs k_b and rotational springs K_b , masses m_b and spring-mass system which consists of a spring of stiffness k_z and a mass m_z . Both ends of beam are connected with linear

springs and rotation springs, which are k_0, K_0 at $x = 0$ and k_L, K_L at $x = L$. The governing differential equation of the free vibration of the beam can be written as:

$$EI \frac{\partial^4 W}{\partial x^4} - \rho s \omega^2 W + \left(\sum_{i=1}^{Nk} Uk_i(x)k_{b,i} + \sum_{i=1}^{Nc} Uc_i(x)k_{e,i} \right) W - \omega^2 \sum_{i=1}^{Nm} Um_i(x)m_{b,i}W(x) - \sum_{i=1}^{NK} K_i \frac{\partial Uk_i(x)}{\partial x} \frac{\partial W(x)}{\partial x} = 0 \tag{1}$$

where $W(x)$ is the flexural displacement and ω is the angular frequency; $k_{b,i}$ and $K_{b,i}$ are the stiffness of the linear and rotational springs, respectively distributing at $Uk_i(x)$ and $UK_i(x)$; $m_{b,i}$ is the distributed mass located at $Um_i(x)$; the spring-mass system including mass $m_{z,i}$ and stiffness $k_{z,i}$, which is fixed at $Uc_i(x)$, are equal to a stiffness $k_{e,i}$ [7]:

$$k_{e,i} = k_{z,i} \frac{\omega^2}{\omega^2 - k_{z,i} / m_{z,i}} \tag{2}$$

where $Uk_i(x), UK_i(x), Um_i(x), Uc_i(x)$ are the function of location of attachments varying with length of beam; i presents the i th attachment in order; Nk, NK, Nm and Nc are the total number of the linear springs, rotational springs, masses and spring-mass systems attached to the beam, separately.

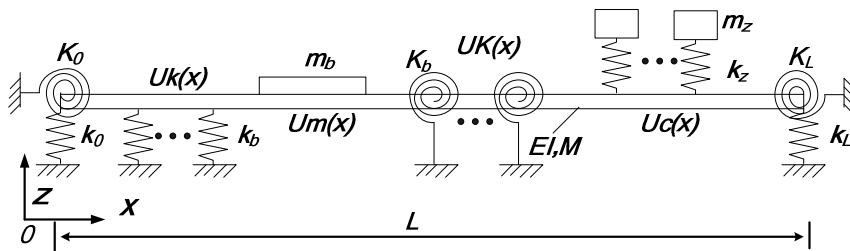


Fig. 1. Skip of beam carrying any type of distributed attachments

The last term on the left side of Eq. (1) is due to the moments applied by the rotational springs and can be derived by considering each of the moments as a pair of closely spaced forces of equal amplitude and opposite directions [20].

The boundary conditions along the elastically restrained edges can be expressed as:

$$\left\{ \begin{aligned} \bar{k}_0 W|_{x=0} &= -W''|_{x=0} \\ \bar{K}_0 W'|_{x=0} &= W'|_{x=0} \end{aligned} \right\}, \left\{ \begin{aligned} \bar{k}_l W|_{x=L} &= W''|_{x=L} \\ \bar{K}_l W'|_{x=L} &= -W'|_{x=L} \end{aligned} \right\}, \left\{ \begin{aligned} \bar{k}_0 &= \frac{k_0}{EI} \\ \bar{K}_0 &= \frac{K_0}{EI} \end{aligned} \right\}, \left\{ \begin{aligned} \bar{k}_L &= \frac{k_L}{EI} \\ \bar{K}_L &= \frac{K_L}{EI} \end{aligned} \right\} \tag{3}$$

A general boundary conditions (such as simply supported, clamped, free and so on) are described in the Eq.(3), which can be directly obtained by accordingly choosing the spring stiffness to be an extremely large or small number.

As a consequence, the Rayleigh-Ritz method or some other methods have been usually used to find an approximate solution. But in the study, to obtain an exact solution, the displacement function will be sought in the form of series expansions as:

$$W(x) = \sum_{n=0}^{\infty} A_n \cos \lambda_n x + \sum_{l=1}^4 c^l \xi^l(x), \quad (\lambda_n = n\pi / L \quad 0 \leq x \leq L) \tag{4}$$

where c^l are coefficients of displacement function and $\xi^l(x)$ represent a set of closed-form sufficiently smooth functions defined over $[0, L]$, which are described as follows [20]:

$$\xi^l(x) = s_1^l \sin\left(\frac{\pi x}{2L}\right) + s_2^l \cos\left(\frac{\pi x}{2L}\right) + s_3^l \sin\left(\frac{3\pi x}{2L}\right) + s_4^l \cos\left(\frac{3\pi x}{2L}\right), \quad l = 1, \dots, 4 \quad (5)$$

where:

$$\left\{ \begin{array}{l} s_1^1 = \frac{9L}{4\pi} \\ s_2^1 = 0 \\ s_3^1 = -\frac{L}{12\pi} \\ s_4^1 = 0 \end{array} \right\}, \left\{ \begin{array}{l} s_1^2 = 0 \\ s_2^2 = -\frac{9L}{4\pi} \\ s_3^2 = 0 \\ s_4^2 = -\frac{L}{12\pi} \end{array} \right\}, \left\{ \begin{array}{l} s_1^3 = \frac{L^3}{\pi^3} \\ s_2^3 = 0 \\ s_3^3 = -\frac{L^3}{3\pi^3} \\ s_4^3 = 0 \end{array} \right\}, \left\{ \begin{array}{l} s_1^4 = 0 \\ s_2^4 = -\frac{L^3}{\pi^3} \\ s_3^4 = 0 \\ s_4^4 = -\frac{L^3}{3\pi^3} \end{array} \right\}$$

In [20], it had been proven mathematically that the series expansion given in Eq. (5) is able to expand and uniformly converge to any function. Mathematically, the displacement function, which represents the exact solution, has to satisfy the governing equation at every filed and the boundary conditions at every boundary point. So our attention will be directed to solving the unknown expansion coefficients by letting the assumed solution satisfy both the governing equations and boundary conditions.

Substituting the displacement expression Eq. (5) into the boundary condition Eq. (3), results are shown as:

$$\left\{ \begin{array}{l} \sum_{l=1}^4 [c^l (\overline{k_0} \xi^l(0) + \xi^{n_l}(0))] = -\overline{k_0} \sum_{n=0}^{\infty} A_n \\ \sum_{l=1}^4 [c^l (\overline{K_0} \xi^l(0) - \xi^{n_l}(0))] = -\lambda_n^2 \sum_{n=0}^{\infty} A_n \\ \sum_{l=1}^4 [c^l (\overline{k_L} \xi^l(L) - \xi^{m_l}(L))] = -\overline{k_L} \sum_{n=0}^{\infty} A_n (-1)^n \\ \sum_{l=1}^4 [c^l (\overline{K_L} \xi^l(L) + \xi^{m_l}(L))] = \lambda_n^2 \sum_{n=0}^{\infty} A_n (-1)^n \end{array} \right. \quad (6)$$

It is clearly shown that the satisfaction of these constraint equations by the expansion coefficients is equivalent to an exact satisfaction of all the boundary conditions form above equations.

Written in matrix form:

$$Hc = Qa \quad (7)$$

where $c = [c^1, c^2, c^3, c^4]$, $a = [A_0, A_1, \dots, A_n]$, $H = [e^1, e^2, e^3, e^4]^T$, $Q = [f^1, f^2, f^3, f^4]^T$, e^1, e^2, e^3, e^4 are 1×4 matrix, f^1, f^2, f^3, f^4 are $1 \times N$ matrix, respectively, whose elements can be described as:

$$\left\{ \begin{array}{l} e_m^1 = \overline{k_0} \xi^m(0) + \xi^{m_m}(0) \\ e_m^2 = \overline{K_0} \xi^{m'}(0) - \xi^{m_m}(0) \\ e_m^3 = \overline{k_L} \xi^m(L) - \xi^{m_m}(L) \\ e_m^4 = \overline{K_L} \xi^{m'}(L) - \xi^{m_m}(L) \end{array} \right\}, \left\{ \begin{array}{l} f_d^1 = -\overline{k_0} \\ f_d^2 = -\lambda_d^2 \\ f_d^3 = -\overline{k_L} (-1)^d \\ f_d^4 = \lambda_d^2 (-1)^d \end{array} \right\}, \quad m = 1, \dots, 4, \quad d = 1, \dots, N \quad (8)$$

In Eq.(7), it is assumed that all the series expansions are truncated to N for the sake of numerical implementation.

To meet the other condition of exact solution, the displacement function is substituted into the governing differential equation, one is able to obtain:

$$EI\left(\sum_{n=0}^{\infty} \lambda_n^4 A_n \cos \lambda_n x + \sum_{l=1}^4 c^l \frac{d^4 \xi^l(x)}{dx^4}\right) + \left(\sum_{i=1}^{Nk} U k_i(x) k_{b,i} + \sum_{i=1}^{Nc} U c_i(x) k_{e,i} - \omega^2 \sum_{i=1}^{Nm} U m_i(x) m_{b,i}\right) - \rho s \omega^2 \left(\sum_{n=0}^{\infty} A_n \cos \lambda_n x + \sum_{l=1}^4 c^l \xi^l(x)\right) - \sum_{i=1}^{NK} K_i \frac{\partial U K_i(x)}{\partial x} \left(-\sum_{n=0}^{\infty} A_n \lambda_n \sin \lambda_n x + \sum_{l=1}^4 c^l \frac{d \xi^l(x)}{dx}\right) = 0 \tag{9}$$

where $\sum_{l=1}^4 \frac{d^4 \xi^l(x)}{dx^4}$, $\sum_{l=1}^4 \frac{d \xi^l(x)}{dx}$ and $\sum_{l=1}^4 \xi^l(x)$ in the above equation are expanded into Fourier cosine series, Eq. (9) can be transformed to:

$$EI \sum_{n=0}^{\infty} (\lambda_n^4 A_n + \sum_{l=1}^4 c^l \alpha_n^l) \cos \lambda_n x + \left(\sum_{i=1}^{Nk} U k_i(x) k_{b,i} + \sum_{i=1}^{Nc} U c_i(x) k_{e,i} - \omega^2 \sum_{i=1}^{Nm} U m_i(x) m_{b,i} - \rho s \omega^2\right) \sum_{n=0}^{\infty} (A_n + \sum_{l=1}^4 c^l \alpha_n^l) \cos \lambda_n x - \sum_{i=1}^{NK} K_i \frac{\partial U K_i(x)}{\partial x} \left(-\sum_{n=0}^{\infty} A_n \lambda_n \sin \lambda_n x + \sum_{l=1}^4 c^l \alpha_n^l \cos \lambda_n x\right) = 0 \tag{10}$$

The coefficients of Fourier expansion α_n^l , α_n^{-l} , $\alpha_n^{\bar{l}}$ based in [20] are demonstrated as:

$$\alpha_n^l = \sum_{p=1}^4 s_p^l \tau_n^p, \quad \alpha_n^{-l} = \sum_{p=1}^4 s_p^l \sigma_n^p(\lambda_p), \quad \alpha_n^{\bar{l}} = \sum_{p=1}^4 s_p^l (\lambda_p)^4 \tau_n^p \tag{11}$$

By multiplying both sides of Eq. (10) by $\cos \lambda_n x$ ($n = 0, \dots, N - 1$) respectively and integrating from 0 to L , N equations are derived and provided to solve coefficients A_n and natural frequencies of the beam. All of the equations mentioned above can be written in matrix form as:

$$EIJ(\overline{K}a + Bc) - \rho s J \omega^2 (\overline{M}a + Fc) + \left(\sum_{i=1}^{Nk} k b_i k_{b,i} + \sum_{i=1}^{Nc} k e_i k_{e,i} - \omega^2 \sum_{i=1}^{Nm} m b_i m_{b,i} - \sum_{i=1}^{NK} K b_i K_i\right) a = 0 \tag{12}$$

Obviously, Eqs. (7) and (12) cannot be directly combined together to form a characteristic equation of the coefficient vectors a and c . By following the approach traditionally used for determining eigenvalues, one may first solve Eq. (12) for a in terms of c and substitute Eq. (7) into Eq. (12), which will lead to a set of homogeneous equations:

$$[EIJK_z - \rho s J \omega^2 M_z + \left(\sum_{i=1}^{Nk} k b_i k_{b,i} + \sum_{i=1}^{Nc} k e_i k_{e,i} - \omega^2 \sum_{i=1}^{Nm} m b_i m_{b,i} - \sum_{i=1}^{NK} K b_i K_i\right)] a = 0 \tag{13}$$

where $K_z = \overline{K} + BH^{-1}Q$, $M_z = \overline{M} + FH^{-1}Q$.

Eq. (13) gives the matrix characteristic equation from which all the eigen pairs can be easily determined. If attachments include stiffness-mass system, the matrix becomes nonlinear. Its eigenvalue can be solved by transforming the problem to solution of polynomial matrix eigenvalue through linearization [21]. Once the eigenvector a is determined for a given eigenvalue, the corresponding vector c can be calculated directly by using Eq. (7).

Subsequently, the mode shapes can be constructed by substituting a and c into Eq. (4). If a load vector is added to the right side of Eq. (10), forced vibration can be also determined.

3 Numerical Results

In the following discussions, the study of the dynamic performance of the free vibration of beams fitted with attachments is divided into two groups. One group is focused on the beam carrying point attachments; the other is on that of beam holding distributed attachments. Third, the proposed method is expanded to solve the natural frequency of the free vibration of beam on Pasternak soil. At last, the results obtained using proposed methods are compared with the results obtained by WFEM.

3.1 Beams fitted with point attachments

To validate the accuracy of the proposed method, a 1 m long beam was considered. The investigation is based on the material presented in [10].

Firstly, the case of beam carrying any number of lumped mass is considered. In order to compare expediently, the locations and weights of masses of the beam are taken as those in [10]. So, only two lump masses are connected to beam, and the ends are all clamped. In this situation, $Uk_i(x)$, $UK_i(x)$, $Uc_i(x)$, $k_{b,i}$, $K_{b,i}$ and ke_i are set to 0 and $k_0 = k_L = K_0 = K_L = 1 \times 10^{11}$ in Eq. (3).

The computed natural frequency coefficients $\beta_i = L\sqrt{\omega_i\sqrt{\rho s / EI}}$ of the beam with two lumped masses obtained by the proposed method are compared with the results obtained in [10] are presented in Table 1. A good agreement is found between the results of the proposed analytic method in the paper and the reference results.

Table 1. The natural frequency coefficients β_i for a clamped-clamped beam carrying two concentrated masses

Location	Method in this paper	Ref. [10]
$Um_1(x) = \delta(x-L/4) \quad Um_2(x) = \delta(x-3L/4)$		
$m_{b,1} = m_{b,2} = 0.1\rho sL$	4.5669	4.566751893
	7.1931	7.191050372
	10.2398	10.2346377
$m_{b,1} = m_{b,2} = 2\rho sL$	3.3053	3.305275348
	4.4575	4.457367357
	8.4692	8.466713346
$Um_1(x) = \delta(x-L/3) \quad Um_1(x) = \delta(x-2L/3)$		
$m_{b,1} = m_{b,2} = 0.01\rho sL$	4.6946	4.694597403
	7.7743	7.774266802
	10.9874	10.98735139
$m_{b,1} = m_{b,2} = \rho sL$	3.3281	3.327923388
	5.1759	5.173572207
	10.7604	10.75788317

Secondly, the type of attachments is changed from mass to stiffness-mass system and the boundary condition is modified to one end free and the other is clamped. Accordingly, $UK_i(x)$, $Um_i(x)$, $Uk_i(x)$, kb_i , Kb_i , mb_i are set to 0 and $k_0 = K_0 = 1 \times 10^{11}$, $k_L = K_L = 0$. The dimensional and physical properties for the cantilever beam are the same as the

configuration previously studied in [15]. The results are shown in Table 2 for beam carrying one spring-mass system at two different locations respectively: $x_1 = 0.75L$ and $x_1 = L$.

Table 2. The natural frequency coefficients β_i for a cantilever beam carrying a spring-mass system

Location	Method in this paper	Reference [15]
$Uc_1(x) = \delta(x - 3L / 4)$		
$k_{z,1} = 3k_a \quad m_{z,1} = 0.2m_a$	1.6481	1.6489
	2.2406	2.2433
	4.7047	4.7002
	7.8592	7.8642
	10.997	11.01
$Uc_1(x) = \delta(x - L)$		
$k_{z,1} = 100k_a \quad m_{z,1} = 0.5m_a$	1.4131	1.4168
	3.8908	3.8973
	5.7619	5.7694
	8.103	8.109
	11.061	11.09

To make comparison more extensively, the lowest five natural frequency coefficients of cantilever beam carrying three spring masses located at $x_1 = 0.1L$, $x_2 = 0.4L$ and $x_3 = 0.8L$ are highlighted in Table 3. The current results match well with those obtained using other techniques in all these cases.

Table 3. The natural frequency coefficient β_i for a cantilever beam carrying three spring-mass systems

Location	Method in this paper	Reference [15]
$Uc_1(x) = \delta(x-0.1L) \quad Uc_2(x) = \delta(x-0.4L) \quad Uc_3(x) = \delta(x-0.8L)$		
$k_{z,1} = 3k_a \quad m_{z,1} = 0.2m_a$	1.2651	1.2651
$k_{z,2} = 4.5k_b \quad m_{z,2} = 0.5m_a$	1.7139	1.7142
$k_{z,3} = 6k_a \quad m_{z,3} = m_a$	1.9678	1.9676
	2.3317	2.3314
	4.7152	4.7153

3.2 Beams carrying distributed attachments

Although the beam carrying lumped attachments is the special situation of the beam carrying distributed attachments when the Dirac delta function $\delta(x)$ is considered as a distribution function, it is not fully enough to prove that the current method could be used to solve the problem of a beam fixed with distributed attachments. So, a simply–simply supported uniform beam with up to three segments of uniformly distributed spring-mass will be investigated in detail. The mass and stiffness of the spring-mass are constants in each segment, but vary from one segment to another. There are three distribution functions which can be written commonly as:

$$H_j(r_j^2 - (x - c_j)^2) = \begin{cases} 1 & x_j^1 \leq x \leq x_j^2 \\ 0 & x < x_j^1, x > x_j^2 \end{cases} \quad (14)$$

where $r_j = (x_j^2 - x_j^1) / 2$, $c_j = (x_j^2 + x_j^1) / 2$, x_j^1 , x_j^2 are lower and higher limits of the j th segment respectively.

The structural parameters of simply–simply supported beam are set as below: $s = 2 \times 10^{-4} \text{ m}^2$, $I = bh^3 / 12 = 1.6667 \times 10^{-9} \text{ m}^4$, $m_d = 1.56 \text{ kg / m}$, $k_d = EI / L^4 = 3.5 \times 10^2 \text{ N / m}^2$,

$K_0 = K_L = 0$, $k_0 = k_L = 1 \times 10^{11}$, m_d and k_d are dimensionless parameters used as basement of $m_{z,i}$ and $k_{z,i}$ in the following discussions.

The spring-mass occupies the full length of the beam and the lengths of the three segments are the same. Accordingly, the distribution functions are: $H_1(1/36 - (x-1/6)^2)$, $H_2(1/36 - (x-1/2)^2)$ and $H_3(1/36 - (x-5/6)^2)$, respectively.

The spring-masses have the parameters $k_{z,1} = 500k_d$ and $m_{z,1} = 2.5m_d$ on the first segment, $k_{z,2} = 500k_d$ and $m_{z,2} = 5m_d$ on the second, $k_{z,3} = 500k_d$ and $m_{z,3} = 10m_d$ on the third. As the dimensionless natural frequencies in [7] are the square of natural frequency coefficients β_i in this paper, results are extracted from the reference and filled in Table 4, in which the excellent agreement is achieved by comparing outcomes from the proposed method and reference.

Table 4. The groups of dimensionless natural frequencies coefficients β_i for a simply-simply supported beam with three segments of uniformly distributed stiffness-masses system

Order of group	First (reference [7])	Second (reference [7])	Third (reference [7])	Fourth (reference [7])
1	1.8857 (1.8858)	2.7174 (2.717)	3.5321 (3.5318)	5.1333 (5.1333)
2	2.6391 (2.6391)	3.1106 (3.111)	3.7465 (3.7465)	6.7617 (6.7612)
3	2.6575 (2.6575)	3.1575 (3.1575)	3.7587 (3.7587)	9.5759 (9.5727)
4	2.6589 (2.6588)	3.1613 (3.1613)	3.7603 (3.7601)	12.643 (12.629)
5	2.659 (2.6591)	3.162 (3.162)	3.7605 (3.7604)	15.763 (15.74)
6	2.6591 (2.6591)	3.1622 (3.1623)	3.7606 (3.7605)	18.844 (18.868)

The attachment distribution function used above isn't various with length of beam at every segment, it is a constant in short.

Now, let us consider a few more complicated problems. The beam is divided into two segments by a point x_v . At its left, the density of spring-masses system arises with length of beam and the opposite situation happens at right. Accordingly, the distribution functions are defined respectively as $V_1(x)$ and $V_2(x)$:

$$\begin{cases} V_1(x) = \frac{2}{x_v} x & 0 \leq x \leq x_v \\ V_2(x) = -\frac{2}{(L-x_v)}(x-x_v) + 2 & x_v < x \leq L \end{cases} \quad (15)$$

where x_v is the point dividing beam into different parts.

In analysis, the locations of x_v are $L/4$, $L/3$ and $L/2$ respectively. The spring-masses have the parameters $k_{z,1} = k_{z,2} = 500k_d$ and $m_{z,1} = m_{z,2} = 5m_d$ on the first and second segments. The lowest four groups of natural frequencies β_i are calculated in Table 5. If parameters of spring-masses are changed to $k_{z,1} = k_{z,2} = 500k_d$ and $m_{z,1} = m_{z,2} = 2.5m_d$, the natural frequencies coefficients β_i are obtained in Table 6.

It is clearly shown that the location of dividing point x_v which has the greatest influence on the first and second natural frequencies of beam are $L/2$ and $L/4$ from results in Tables 5

and 6. Furthermore, it has been demonstrated in [22] that the largest magnitude of the first and second mode shape of simply-simply supported beam without attachments is $L/2$ and $L/4$ respectively, at which the density of distributed spring-masses is highest with distribution functions $V_1(x)$ and $V_2(x)$ at $x_v = L/2$, $x_v = L/4$. That is the reason why the attachments make greatest effect on the natural frequency of beam.

Table 5. The groups of dimensionless natural frequencies coefficients β_i for a simply-simply supported beam with two segments of non-uniformly distributed spring-masses

Order of group	First			Second		
	$L/4$	$L/3$	$L/2$	$L/4$	$L/3$	$L/2$
1	1.8542	1.8348	1.8188	5.3042	5.3779	5.4514
2	2.9314	2.9336	2.9392	6.8308	6.8049	6.7563
3	3.1159	3.1164	3.1157	9.5810	9.4766	9.5837
4	3.1480	3.1497	3.1480	12.6432	12.6444	12.6432

Table 6. The groups of dimensionless natural frequencies coefficients β_i for a simply-simply supported beam with parameters of spring-masses $k_{z,1} = k_{z,2} = 500k_d$ and $m_{z,1} = m_{z,2} = 2.5m_d$

Order of group	First			Second		
	$L/4$	$L/3$	$L/2$	$L/4$	$L/3$	$L/2$
1	2.1485	2.128	2.1109	5.4432	5.5139	5.5857
2	3.4739	3.4769	3.4841	6.8554	6.8286	6.778
3	3.7048	3.7054	3.7046	9.5828	9.5784	9.5856
4	3.7435	3.7434	3.7435	12.644	12.6445	12.644

3.3 Beams on Pasternak soil

This method also can be used to analyze the problem of free vibration of Euler beam on Pasternak soil. According to a beam with flexible ends resting on Pasternak soil in [13], the influences of two parameters elastic soil are equivalent to shear and linear elastic springs, which connect to beam from one end to the other end. The equation of motion of this beam can easily be deduced by means of Hamilton's principle [13]:

$$EI \frac{\partial^4 W}{\partial x^4} - \bar{k}_1 \frac{\partial^2 W(x)}{\partial x^2} + (\bar{k}_0 - \rho s \omega^2) W = 0 \tag{16}$$

where \bar{k}_0 is the Winkler modulus of the subgrade reaction, \bar{k}_1 is the second foundation parameter. Substituting Eq. (4) into Eq. (16), the governing differential function is transformed to:

$$EI \left(\sum_{n=0}^{\infty} \lambda_n^4 A_n \cos \lambda_n x + \sum_{l=1}^4 c^l \frac{d^4 \xi^l(x)}{dx^4} \right) - \bar{k}_1 \left(- \sum_{n=0}^{\infty} A_n \lambda_n^2 \cos \lambda_n x + \sum_{l=1}^4 c^l \frac{d^2 \xi^l(x)}{dx^2} \right) + (\bar{k}_0 - \rho s \omega^2) \left(\sum_{n=0}^{\infty} A_n \cos \lambda_n x + \sum_{l=1}^4 c^l \xi^l(x) \right) = 0 \tag{17}$$

It can be written in matrix form as follows:

$$(EIK_z + \bar{k}_0 M_z - \bar{k}_1 C_z - \rho s \omega^2 M_z) a = 0 \tag{18}$$

The natural frequencies are calculated by solving the eigenvalue problem of governing Eq. (18), while the mode shapes are derived from eigenvector and relationship between a and c in Eq. (7). After changing the right side of Eq. (18) to force vector, forced vibration problem of beam on Pasternak soil can also be solved.

To compare with the results in [13], non-dimensional soil parameter coefficients are set as follows:

$$\Omega_0 = \bar{k}_0 L^4 / EI, \Omega_1 = \bar{k}_1 L^2 / EI \pi^2 \tag{19}$$

The dimensionless natural frequencies β_i obtained by the current method in this paper are compared with calculated in [13] and are presented in Table 7. From the Table 7, the result shows excellent agreement.

Table 7. Numerical comparisons with reference [13] for a clamped-clamped beam on Pasternak soil

$\Omega_0 \backslash \Omega_1$	0	0	1	1	2.5	2.5
	this paper	reference [13]	this paper	reference [13]	this paper	reference [13]
0	4.73	4.73	4.9925	4.994	5.3183	5.32
	7.8531	7.854	8.0774	8.078	8.3808	8.381
	10.995	10.996	11.174	11.174	11.427	11.43
100	4.9504	4.95	5.1823	5.182	5.4772	5.477
	7.9042	7.904	8.1244	8.124	8.423	8.423
	11.014	11.014	11.192	11.192	11.444	11.444
100000	10.123	10.123	10.152	10.152	10.194	10.194
	10.839	10.839	10.927	10.927	11.055	11.055
	12.526	12.526	12.648	12.648	12.825	12.825

The above examples are presented as special cases of beam carrying any type of attachments with elastically restrained boundary. It is shown that the frequency parameters for different type of attachments, distributions and boundary conditions can be accurately determined by modifying the parameters of attachments, the distribution functions and the stiffness of the restraining springs accordingly. It should be emphasized that unlike most existing approaches, the current method provides a unified solution for a variety of attachments, its distributions and boundary conditions. The change from one case to another is as simple as modifying the material properties or structural dimension.

3.4 Compare the damage detection results with the WFEM

In order to illuminate the proposed method ulteriorly. A cantilever beam model was selected as the damage detection model for comparing the radian frequency using proposed method with the results obtained using WFEM. When $K_L, k_L = 0$, it can be seen as cracked cantilever beam which shown in Fig. 2. In the WFEM, the cracked beam is modeled as two segments connected by two massless springs. But the crack is described by using the distribution function of bending stiffness and mass in the proposed method, which is shown in Eq. (20):

$$EIP_1(x) \frac{\partial^4 W}{\partial x^4} - \rho s P_2(x) \omega^2 W = 0 \tag{20}$$

where $P_1(x)$ and $P_2(x)$ present bending rigid distribution function and mass distribution function, respectively, as follows:

$$P_1(x) = \begin{cases} \frac{Ebh^3}{12} & x \neq L_a \\ \frac{Eb(h/2 - h_1)^3}{3} + \frac{Ebh^3}{24} & x = L_a \end{cases} \tag{21}$$

$$P_2(x) = \begin{cases} \rho bh & x \neq L_a \\ \rho b(h - h_1) & x = L_a \end{cases} \tag{22}$$

In the Eq. (21)-Eq. (22), L_a presents the location of the crack, h_1 is the height of the crack. Then, the radian frequencies can be obtained by solving the eigenvalue problem of governing Eq. (20).

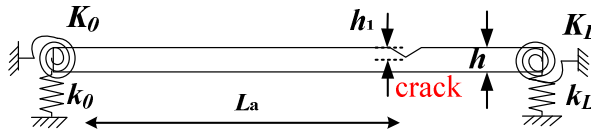


Fig. 2. Cracked beam simulated by spring

For comparing expediently, the material parameters and geometrical parameters of the beam are taken as those in [23]. The values of radian frequencies obtained by the proposed method in the paper are compared with calculated in [23] and are tabulated in Table 8. The result shows excellent agreement in Table 8, respectively. Obviously, the proposed method has a high accuracy like the WFEM in detecting cracks.

Table 8. Comparison of the values of radian frequencies between proposed method and WFEM for different crack cases [23] listed in Table 8

Case	Radian frequency (rad×s ⁻¹)					
	Proposed method			WFEM [23]		
	Mode 1	Mode 2	Mode 3	Mode 1	Mode 2	Mode 3
1	419.7042	2630.2671	7364.8757	419.7025	2630.2769	7364.8817
2	417.0565	2624.2187	7361.7086	417.0567	2624.2815	7362.8567
3	417.8291	2630.1832	7355.2859	417.8255	2630.2479	7356.3200
4	416.7638	2600.8914	7313.5792	416.7787	2600.9474	7313.6602
5	418.1076	2587.3071	7364.9841	418.1101	2587.3721	7365.9964
6	419.5747	2611.8687	7362.3451	419.5719	2611.9494	7162.6880

Conclusions

In this paper, a new proposed analytic method is applied to solve free vibration of beams carrying any type of attachments with arbitrary distributions on elastic boundary supports. Through three different simulation examples, which involve beams attached by masses, spring-masses and on Pasternak soil, respectively, the natural frequency and relevant coefficient are calculated and compared with referred to in the references. The contrasts show great agreement, which give evidences that the proposed method is accurate for analyzing the free vibration of the beam carrying attachments by any way.

The study of the dynamic characteristic of free vibration of beams carrying any type attachments with arbitrary distributions on elastic boundary supports by using the analytical method in the paper is important, The proposed analytic method presented in this study not only can be applied to deal with problems in the references. Moreover, its can be extended to analyze free vibration of the beam fixed with attachments which distributes various with its length and the forced vibration analysis.

It is also verified, in the paper, that the radian frequency obtained by the present proposed method can be utilized to detect crack location through comparing the results available by using wavelet finite element method. And the proposed method can be used to detect cracks.

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