828. Piecewise exact solution of nonlinear momentum conservation equation with unconditional stability for time increment

Changhwan Jang[1]**, Hyoseob Kim**[2]**, Sokhwan Choi**[3]**, Jinho Kim**[4]
[1] Korean Intellectual Property Office, Daejeon, Korea
[2, 3] Kookmin University, Seoul, Korea
[4] Yeungnam University, Kyungsan, Korea
E-mail: [1]*cjang@kipo.go.kr*, [2]*hkim@kookmin.ac.kr*, [3]*shchoi@kookmin.ac.kr*, [4]*jinho@ynu.ac.kr*

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Abstract. Exact solution is adopted for computation of the inviscid Burgers equation on finite difference grid. Initial condition and following computed values of the independent variable are assumed to be piecewisely linear between fixed grid points, and local exact solution is used to find the value at the next time step at each grid point. Comparisons of Piecewise Exact Solution Method (PESM), existing upwind scheme, and the analytic solution show that the present method is more accurate than the upwind scheme. The unconditional stability is a strong merit of this method and is shown with a test result.

Keywords: inviscid Burgers equation, piecewise exact solution method, upwind scheme, unconditional stability.

Introduction

Not only the advection equation but also the advection-diffusion equation has been intensively used to explain the flow of fluids in mathematical, physical, scientific, and engineering problems [1-3]. The advection equation becomes the advection-diffusion equation by adding diffusion terms with diffusion coefficients. Reversely, the advection equation can be thought as a partial equation containing advection phenomena only by cutting off the diffusion, or separating the partial differential operator into two, and getting two equations, i.e. the advection equation and the diffusion equation (fractional step method [4, 5] or operator-splitting method [6]). The advection equation often describes transport of a scalar property, e.g. concentration, temperature etc. A very specific case of the advection equation is the Burgers equation, when the dependent variable is the velocity itself [7, 8] instead of an arbitrary scalar property. The advection equation has been a focus of numerical modeling due to possible numerical diffusion or wiggles. Calculation of the inviscid Burgers equation is concerned here. Even though the Burgers equation is nonlinear, and different from linear advection equation, people have used similar numerical methods to solve the Burgers equation.

Fundamentally, the advection-diffusion equation:

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = \nu\frac{\partial^2 u}{\partial x^2} \tag{1}$$

is reduced to the advection equation:

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0 \tag{2}$$

The inviscid Burgers equation is a specific form of the advection equation, when $a = u$.

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0 \tag{3}$$

The same numerical schemes for linear advection equation can also be applied to the above nonlinear advection equation but the velocity should be represented by a finite difference value. The nonlinear Burgers equation also influences the stability problem.

The inviscid Burgers equation is useful because it has a close link with the conservation laws. The inviscid Burgers equation is an example of the nonlinear momentum conservation equation. Solutions for the inviscid Burgers equation can be obtained on spatially fixed grid by using existing numerical methods [9, 10]. The inviscid Burgers equation can be written in conservative form as:

$$\frac{\partial u}{\partial t} + \frac{\partial F(u)}{\partial x} = 0 \tag{4}$$

$$F(u) = \frac{u^2}{2} \tag{5}$$

where x is the spatial axis, t is the time, and u is the velocity in the x axis. The velocity field, $u(x,t)$, at the initial point ($t = 0$), is $u(x,0)$, and is called $f(x_0)$. The analytic solution of Equation (4) for the given initial condition is:

$$u_L(x_0,t) = u(x_0,0) = f(x_0) \tag{6}$$

where u_L is the velocity for Lagrangean position:

$$x = x_0 + u_L \cdot t = x_0 + f(x_0)\cdot t \tag{7}$$

$$u(x,t) = u(x_0 + f(x_0)\cdot t, t) = u_L(x_0,t) = u_L(x_0,0) = f(x_0) \tag{8}$$

The problem is that it is not always possible to get the solution in an explicit form for an arbitrary function, f. One of frequently used numerical schemes to solve the advection equation is the upwind scheme [9]:

$$\frac{u_i^{n+1} + u_i^n}{\Delta t} = -\frac{\overline{u}_i^n}{\Delta x}(u_{i+1}^n - u_i^n \text{ or } u_i^n - u_{i-1}^n) \tag{9}$$

where $\overline{u}_i^n$ is a representative velocity at number i, which could be u_i^n, u_{i+1}^n, u_{i-1}^n, $(u_i^n + u_{i+1}^n)/2$, $(u_{i-1}^n + u_i^n)/2$. If we use the difference equation for the Burgers equation in the conservative form, the difference equation becomes:

$$\frac{u_i^n}{\Delta t} = -\frac{1}{2\Delta x}\left\{(u_{i+1}^n)^2 - (u_i^n)^2, \text{or} (u_i^n)^2 - (u_{i-1}^n)^2\right\} \tag{10}$$

then $(u_i^n + u_{i+1}^n)/2$ or $(u_{i-1}^n + u_i^n)/2$ corresponds to the representative velocity.

Similar the above upwind scheme, the Lax-Wendroff scheme [13], the modified Lax-Wendroff scheme by Roe [15], TVD [16], the box scheme [17], or the Leap-frog scheme [18] adopt different numerical schemes which are useful for the linear advection equation, but not for the Burgers equation in a sense that the selection of the representative velocity value produces small or large numerical errors.

There have been trials to make use of the exact solution for fixed grid system. A Lagrangean method, MAC, introduces moving particles which conserve physical properties, but requires cumbersome treatment of the density of the particles in a fixed grid space, which is a defect of the method [11]. Here we propose a new method to make use of the exact solution of the inviscid Burger's equation. The shock waves can be generated while solving the Burgers equation depending on conditions [14].

Piecewise Exact Solution Method

If the dependent variable is initially distributed linearly in space then we find an exact solution which is:

$$u(x,t_0) = dx + e \tag{11}$$

$$u = \frac{x+c}{t} \tag{12}$$

where c is a coefficient, $c = e/d$. Equation (11) satisfies the inviscid Burgers equation (4).

An interesting feature of this solution is that the initial time, t_0, cannot be arbitrarily provided, but a fixed value, $1/d$, should be assigned. If the starting time is negative, and time proceeds, then we encounter a singularity at $t = 0$ which means a "shock", as time.

If two boundary value sets of x and u are given, Equation (12) can be used for both accelerating and decelerating phases by using either positive or negative domain for x or t. Given the boundary values at two adjacent grid points, we can find the solution at new time step on a grid point by using above the exact solution.

For positive, spatially increasing velocity in the positive x axis, for instance, Equation (12) can be expressed in a numerical form as:

$$u_i^n = \frac{x_i + c}{t}; \quad u_{i-1}^n = \frac{x_i - \Delta x + c}{t}; \quad u_i^{n+1} = \frac{x_i + c}{t + \Delta t} \tag{13}$$

where Δx is the spatial increment, Δt is the temporal increment, the subscript i or $i-1$ is the number of grid point, and the superscript n or $n+1$ is the number of time step.

Dropping out x_i and t, the exact velocity value at node i at the new time step becomes (Figure 1):

$$u_i^{n+1} = \frac{\Delta x}{(u_i^n - u_{i-1}^n)\Delta t + \Delta x} u_i^n \tag{14}$$

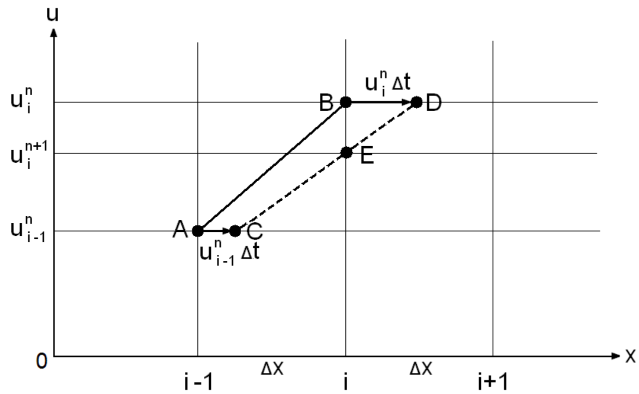

Fig. 1. Piecewise exact solution for positive increasing velocity distribution

If the front side velocity times temporal increment (Δt) exceeds m times the spatial increment (Δx), see the segment of $A(x_{i-m-1}, u_{i-m-1}^n)$ and $B(x_{i-m}, u_{i-m}^n)$ in Figure 2, we obtain the following exact solution:

$$u_i^{n+1} = \frac{\Delta x}{(u_{i-m}^n - u_{i-m-1}^n)\Delta t + \Delta x}\left\{(m+1)\cdot u_{i-m}^n - m\cdot u_{i-m-1}^n\right\},$$

for $m\Delta x \le u_{i-m}^n \Delta t < (m+1)\Delta x, \quad m = 1, 2, 3, \cdots$ (15)

Equation (15) also satisfies Equation (14) for $m = 0$, thus its use can be extended for $m = 0, 1, 2, \cdots$.

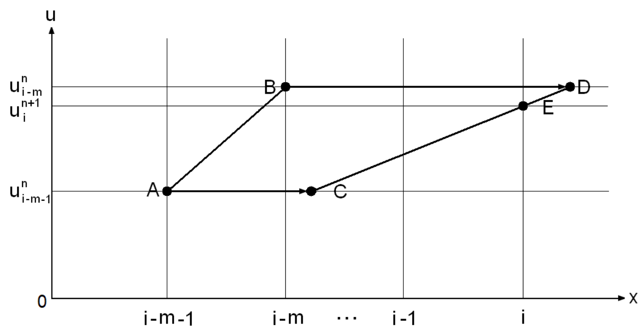

Fig. 2. Piecewise exact solution for large temporal increment ($m \ge 1$)

For positive, spatially decreasing velocity distribution in the x axis, the velocity at grid point i at new time step can be calculated from the same equation, Equation (14), see Figure 3. This solution has similarity to existing upwind schemes in which upstream values at previous time step are used for calculation of the velocities at the new time step.

For negative velocities (Figure 4), the following equation is used:

$$u_i^{n+1} = \frac{\Delta x}{(u_{i+1}^n - u_i^n)\Delta t + \Delta x} u_i^n \tag{16}$$

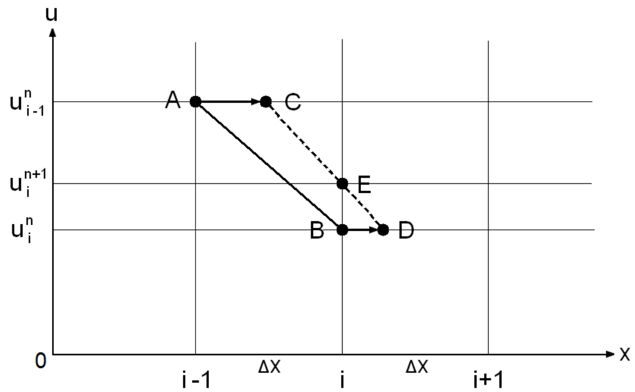

Fig. 3. Piecewise exact solution for positive decreasing velocity distribution

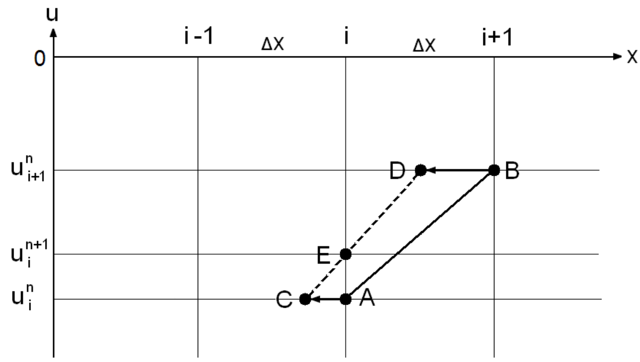

Fig. 4. Piecewise exact solution for negative increasing velocity distribution

When the two neighboring velocities, u_{i-1}^n and u_i^n, are in the opposite direction (Figures 5 and 6), one case is that velocity increases in the x axis:

$$u_i^{n+1} = \frac{\Delta x}{\left|u_i - u_{i-1}\right|\Delta t + \Delta x} u_i^n \tag{17}$$

$$u_{i-1}^{n+1} = \frac{\Delta x}{\left|u_i - u_{i-1}\right|\Delta t + \Delta x} u_{i-1} \tag{18}$$

If the two adjacent velocity values have opposite signs, but velocity decreases in the x axis, then Equations (14) and (16) are used.

The present method remains valid unless the spatial velocity gradient becomes infinity. The infinite velocity gradient corresponds to the singularity of the function F, and consequently solution cannot be obtained numerically from the singularity. Physical similarity exist e.g. shock waves, wave breaking, or hydraulic jump.

3. Results and discussion

The present method is using piecewise exact solution for each segment of the given arbitrary distribution of velocity, and shortly called as PESM afterwards. If the given function is a linear function in a wide computation domain, the solution remains as a linear function as time proceeds. In order to examine its accuracy PESM is applied to a case of a linear initial condition for positive x:

$$u(x,0)=x\,,$$
$$\Delta x=1.0\,,\;\;0\le x\le 100\,,$$
$$\Delta t=1.0\,,\;\;0<t\le 25\,.$$

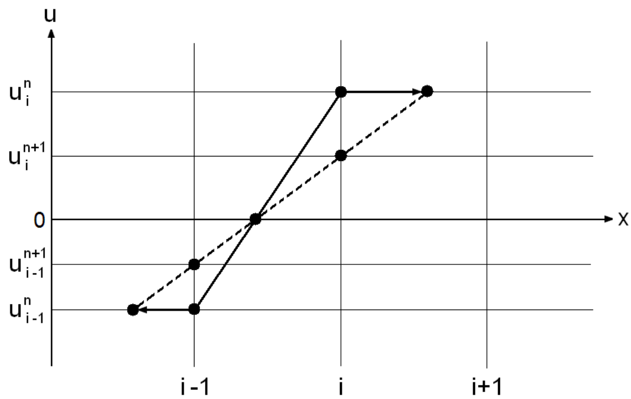

Fig. 5. Piecewise exact solution for flattening velocities of opposite signs

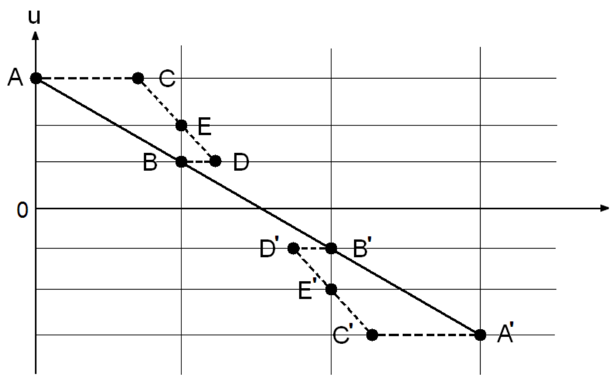

Fig. 6. Piecewise exact solution for steepening velocities of opposite signs

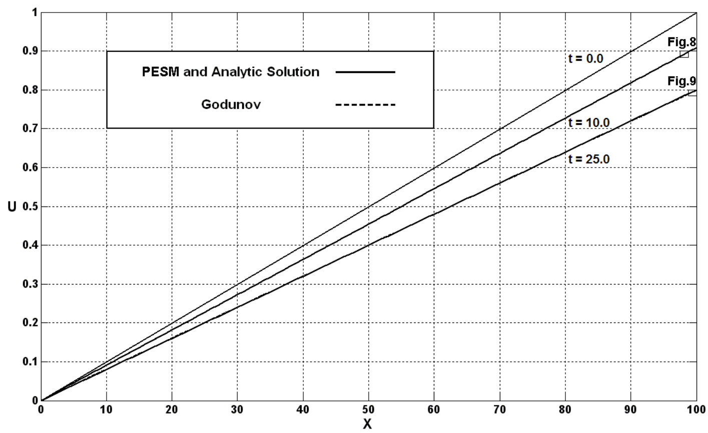

Fig. 7. Test results of PESM and Godunov for case of linear velocity distribution

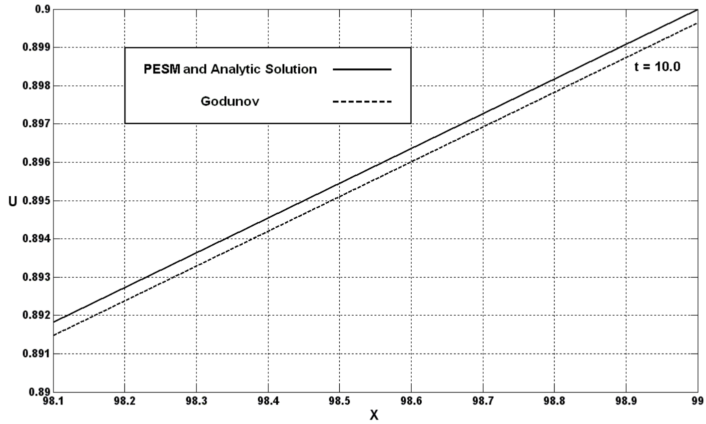

Fig. 8. Zoomed plot of Figure 6 for $t = 10.0$

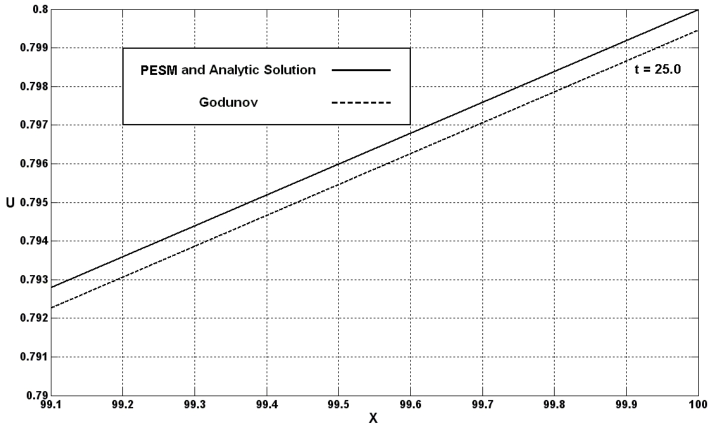

Fig. 9. Zoomed plot of Figure 6 for $t = 25.0$

The results of the present method coincide with the analytic solution throughout the time as expected, because the present method uses piecewise exact solution, and the given function has a single gradient within the whole domain. The results were also compared with the results of an existing scheme [9]. Godunov's scheme is a first-order upwind scheme. The upwind scheme is not free from numerical error at each time step, and accumulates larger total error as time proceeds, see Figs. 7, 8, and 9.

Next, the PESM was applied to an expansion wave composed of three linear functions, constant u_l (left velocity), straight connection line, and constant u_r (right velocity):

$$u(x,0) = u_l = 0.5\,, \qquad x < 25$$

$$u(x,0) = \frac{1}{2}(u_l + u_r) = \frac{1}{4}(x - 26) + \frac{3}{4}\,, \qquad 25 \le x \le 27$$

$$u(x,0) = u_r = 1.0\,, \qquad x > 27$$

$$\Delta x = 1\,,\ \Delta t = \frac{2}{3}\,,\ 0 \le t \le 16\,.$$

Results for both the PESM and Godunov's scheme show similar behavior in a large scale plot, Figure 10. The computed central position of the wave with both the PESM and Godunov's scheme are exact. However Godunov's scheme becomes farther from the analytic solution as time proceeds. Numerical diffusion of the PESM is slightly smaller than that of Godunov's scheme, see Figures 10(a) and 10(b).

Similarly, the PESM and Godunov's scheme were applied to a shock wave case:

$$u(x,0) = u_l = 1.0, \qquad x < 25$$

$$u(x,0) = \frac{1}{2}(u_l + u_r) = \frac{1}{4}(26 - x) + \frac{3}{4}, \qquad 25 \le x \le 27$$

$$u(x,0) = u_r = 0.5, \qquad x > 27$$

$$\Delta x = 1, \ \Delta t = \frac{2}{3}, \ 0 \le t \le 16.$$

The PESM again produces similar shape of computation results to Godunov's first order upwind scheme in a large scale plot (Figure 11). Numerical diffusion of the PESM is slightly smaller than Godunov's scheme, see Figures 11(a) and 11(b). The PESM shows more accurate solution for both spreading and shock waves, which distinguishes itself from the existing first order upwind schemes.

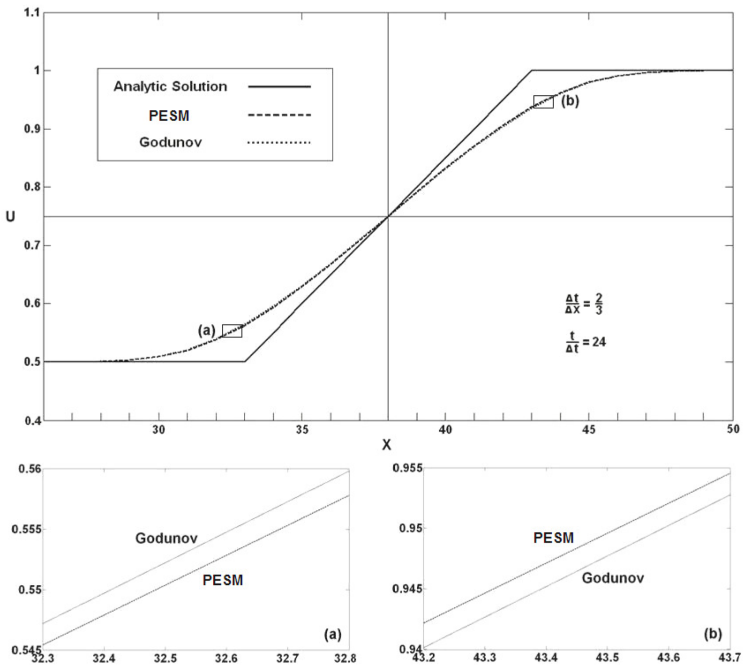

Fig. 10. Calculated results of expansion wave by PESM and Godunov

Initially smooth distribution of the velocity in space can be approximated by piecewise linear functions and relatively smooth variation of their gradients. The PESM was applied to the velocity distribution of a bell-shaped function with a rough resolution:

$$u(x,0) = \exp\left(-\frac{x^2}{160}\right), \ -\infty < x < \infty$$

$$\Delta x = 0.02, \ \Delta t = 0.01 \text{ or } 0.03, \ 0 \le t \le 0.09.$$

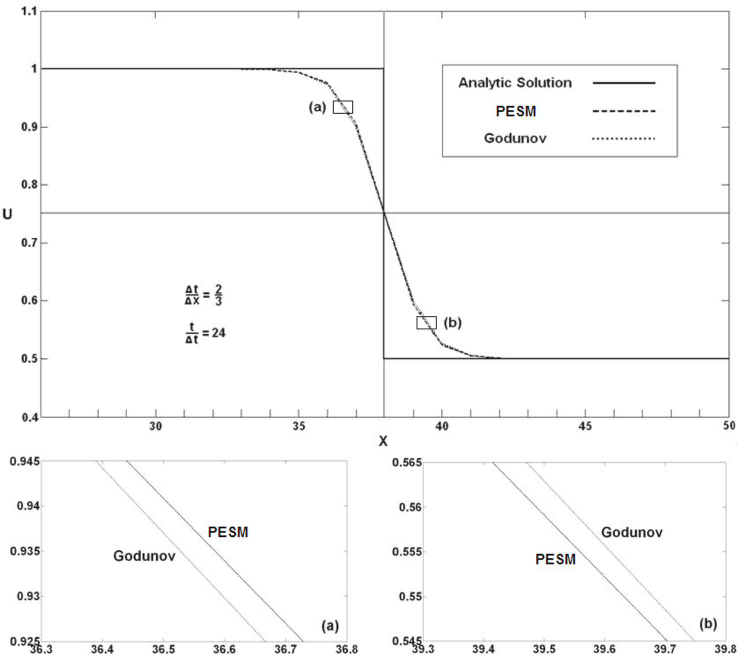

Fig. 11. Calculated results of shock wave by PESM and Godunov

At $t = 0.09$, the computed velocity field for the time increment of 0.01 included maximum total error of 0.125 around the bell crest, which is similar to the maximum total error of 0.131 Godunov's scheme, see Figure 12. Now, an increased time increment, $\Delta t = 0.03$, was applied to the same condition, which does not satisfy the CFL condition. The computed results at time of 0.09 by using the PESM are stable, while those by using Godunov's scheme show the symptom of instability, see Figure 13. This demonstrates the unconditional stability of the PESM with respect to the time increment, unless the problem actually reaches singularity.

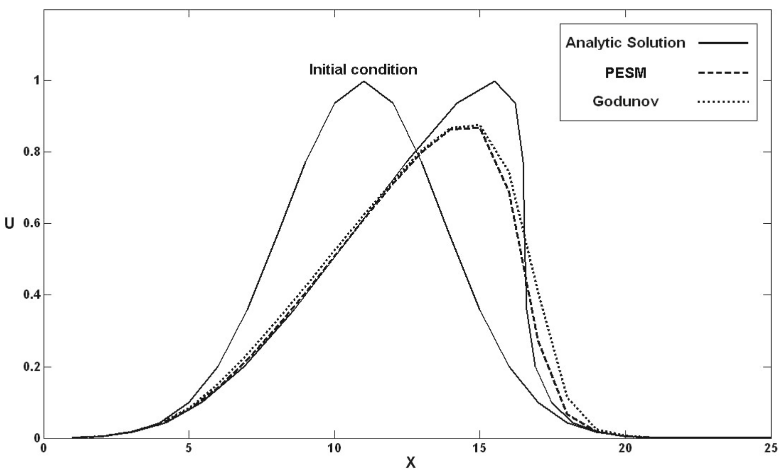

Fig. 12. Comparison with the velocity distribution at time of 0.09 for $\Delta t = 0.01$

It could be said that the PESM gives more accurate solutions than Godunov's first order upwind scheme because of its exactness. Therefore, replacing the first-order finite difference part in many existing schemes with the present method could be recommended here. For example, the existing form of Fromm's scheme [12] is to be used to solve the inviscid Burgers equation, then:

$$u_i^{n+1} = u_i^n - a\frac{\Delta t}{\Delta x}\left(\frac{u_i^n - u_{i-1}^n}{\Delta x}\right) - a\frac{\Delta t}{4\Delta x}\left(1 - a\frac{\Delta t}{\Delta x}\right)\left(\frac{u_{i+1}^n - u_i^n}{\Delta x} - \frac{u_{i-1}^n - u_{i-2}^n}{\Delta x}\right) \tag{19}$$

could be modified into:

$$u_i^{n+1} = \frac{\Delta x}{(u_i^n - u_{i-1}^n)\Delta t + \Delta x}u_i^n - a\frac{\Delta t}{4\Delta x}\left(1 - a\frac{\Delta t}{\Delta x}\right)\left(\frac{u_{i+1}^n - u_i^n}{\Delta x} - \frac{u_{i-1}^n - u_{i-2}^n}{\Delta x}\right) \tag{20}$$

where a is a velocity or characteristic speed in the x-direction, and a discrete value should be assigned if it is used for the Burgers equation.

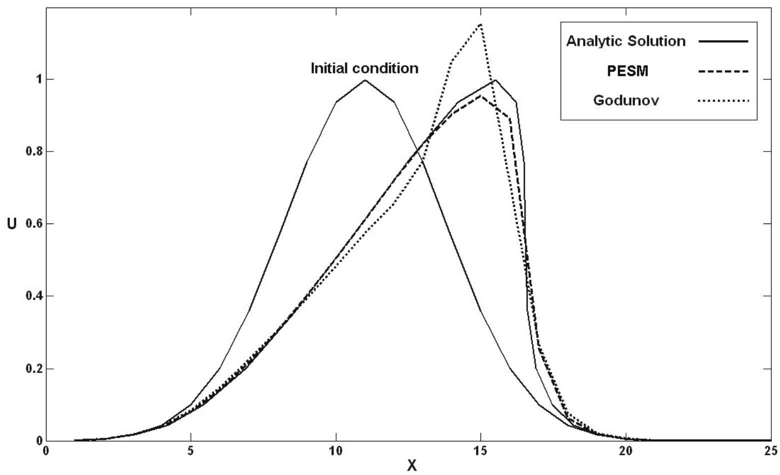

Fig. 13. Comparison with the velocity distribution at time of 0.09 for $\Delta t = 0.03$

And the existing Lax-Wendroff scheme [13]:

$$u_i^{n+1} = u_i^n - a\frac{\Delta t}{\Delta x}\left(\frac{u_i^n - u_{i-1}^n}{\Delta x}\right) - a\frac{\Delta t}{2\Delta x}\left(1 - a\frac{\Delta t}{\Delta x}\right)\left(\frac{u_{i+1}^n - u_i^n}{\Delta x} - \frac{u_i^n - u_{i-1}^n}{\Delta x}\right) \tag{21}$$

could be modified into:

$$u_i^{n+1} = \frac{\Delta x}{(u_i^n - u_{i-1}^n)\Delta t + \Delta x}u_i^n - a\frac{\Delta t}{2\Delta x}\left(1 - a\frac{\Delta t}{\Delta x}\right)\left(\frac{u_{i+1}^n - u_i^n}{\Delta x} - \frac{u_i^n - u_{i-1}^n}{\Delta x}\right) \tag{22}$$

Using Equations (20) and (22) may reproduce more accurate speed of the wave phase than Equation (19) and (21), as was demonstrated in Figures 8 and 9, even through these replacements are not the major interest of this paper.

Conclusions

Accuracy is one of the most important objects during solving the momentum conservation equation. Most existing numerical methods have been developed for the linear advection equation. The PESM is a fairly simple method, and still produces more accurate solutions for the inviscid Burgers equation than Godunov's first order upwind scheme which is adequate for the linear advection equation. The PESM is again an unconditionally stable method for solving the inviscid Burgers equation. If the initial condition is described by continuous piecewise linear functions, the solution can be obtained with arbitrary time increment unless any shock wave develops. The present PESM could be used effectively to solve the momentum conservation equation, especially when it is combined to fractional-step method.

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