677. Nonlinear free vibration analysis of tapered beams by hamiltonian approach

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Abstract. In this paper the Hamiltonian approach is applied to the analysis of the nonlinear free vibration of a tapered beam. The considered problem presents the governing equation of the nonlinear, large-amplitude free vibrations of the beams. The effect of vibration amplitude on the nonlinear frequency is considered. This analytical representation provides excellent approximations to the exact solutions for the whole range of the oscillation amplitudes, reducing the respective error of angular frequency in comparison with the Hamiltonian approach. It is predicted that those methods can find wide application in engineering problems as indicated in this paper.

Keywords: nonlinear vibration, Hamiltonian approach, tapered beams.

Introduction

Many aspects have to be taken into consideration in the design of structures to improve their performance and extend their life, especially in aerospace vehicles, bridges and automobiles. One aspect of the design process is the dynamic response of a structure. The dynamics of distributed parameter and continuous systems, like beams, are governed by linear and nonlinear partial differential equations in space and time.

Kreiger [1] studied the large-amplitude vibration of simply supported beams wherein the governing partial differential equations were reduced to ordinary differential equations and the solution was obtained in terms of elliptic functions using a one-term approximation. Burgreen [2] used the classical continuum approach for the large-amplitude vibration problems of hinged beams. Ritz–Galerkin technique was used to solve the governing nonlinear differential equation of dynamic equilibrium for free and forced vibration of simply supported beams and plates by Srinivasan [3-4]. Evensen [5] extended the study for various boundary conditions using the perturbation method.

A review of the formulations of the 1970s related to structural vibrations was presented by Reddy [6]. Sathyamoorthy [7-8] tried to complete the work on classical methods for the analysis of beams with material, geometric and other types of nonlinearities and also on finite element analysis of nonlinear beams under static and dynamic loads. Mei studied the finite element formulations for large-amplitude vibrations of beams and plates. In all his works the axial deformation was not considered and the average axial force was assumed to be a constant over the element length [9-10].

Singh et al [11] reported a formulation for the nonlinear free vibrations of beams, wherein the dynamic finite element matrix equations were reduced to a scalar equation (using the converged mode shape), which was then solved using direct numerical integration and concluded that the axial displacements cannot be neglected in any nonlinear vibration analysis. Klein [12] used finite element approach and Rayleigh-Ritz for analyzing the vibration of the tapered beams. A dynamic discretization technique was applied to calculate the natural frequencies of a non-rotating double-tapered beam based on both the Euler-Bernoulli and Timoshenko beam theories by Downs [13]. Goorman [14] is given the governing differential equation corresponding to fundamental vibration mode of a tapered beam. Generally, finding an exact or closed-form solution for nonlinear problems is very difficult especially on beam vibrations. Hoseini et al. [15] studied nonlinear vibration behavior of tapered beams using homotopy analysis method. Bayat et al [16] developed the work on the analytical studies on the nonlinear vibration of tapered beams by using max-min approach and homotopy perturbation method (HPM). Shahidi et al [17] used variational approach method and amplitude frequency formulation to obtain an approximate solution for the vibration of tapered beams. In the recent decades many new analytical and numerical approaches have been investigated. The most useful methods for solving nonlinear equations are perturbation methods. They are not valid for strongly nonlinear equations and they have many shortcomings. Many new techniques have appeared in the open literature to overcome the shortcomings of traditional analytical methods such as Energy balance [18-22], Variational approach [23-24], Iteration perturbation [25], the Homotopy perturbation method [26] and other analytical and numerical methods [27-30].

The main objective of this study is to obtain the analytical expression for nonlinear vibration of tapered beams. First, the governing nonlinear partial differential equation using Galerkin method was reduced to a single nonlinear ordinary differential equation. The later equation was solved analytically in time domain using Hamiltonian approach. Finally, Hamiltonian aproach is compared with the exact solution. It can be observed that Hamiltonian approach results are accurate and require smaller computational effort. An excellent accuracy of the Hamiltonian approach (HA) results indicates that those methods can be used for problems in which the strong nonlinearities are taken into account. In dimensionless form, Goorman gives the governing differential equation corresponding to fundamental vibration mode of a tapered beam [14]:

$$\left(\frac{d^2u}{dt^2}\right) + \varepsilon_1\left(u^2\left(\frac{d^2u}{dt^2}\right) + u\left(\frac{du}{dt}\right)^2\right) + u + \varepsilon_1 u^3 = 0 \tag{1}$$

where u is displacement and ε_1 and ε_2 are arbitrary constants. Subject to the following initial conditions:

$$u(0) = A, \qquad \frac{du(0)}{dt} = 0 \tag{2}$$

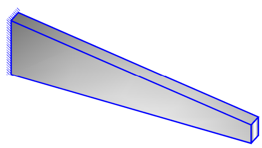

Fig. 1. Schematic representation of a tapered beam

Basic idea of the Hamiltonian approach

Previously, He [31] had introduced the Energy Balance method based on collocation and the Hamiltonian. This approach is very simple but strongly depends upon the chosen location point. Recently, He [32] has proposed the Hamiltonian approach to overcome the shortcomings of the energy balance method. This approach is a kind of energy method with a vast application in conservative oscillatory systems. In order to clarify this approach, consider the following general oscillator:

$$u'' + f(u, u', u'') = 0 \tag{3}$$

with initial conditions:

$$u(0) = A, \qquad u'(0) = 0. \tag{4}$$

Oscillatory systems contain two important physical parameters, i.e. the frequency ω and the amplitude of oscillation A. It is easy to establish a variational principle for Eq. (3), which reads;

$$J(u) = \int_0^{T/4}\left\{-\frac{1}{2}u'^2 + F(u)\right\}dt \tag{5}$$

where T is period of the nonlinear oscillator, $\partial F/\partial u = f$.

In the Eq. (5), $\frac{1}{2}u'^2$ is kinetic energy and *F (u)* potential energy, so the Eq. (5) is the least Lagrangian action, from which we can immediately obtain its Hamiltonian, which reads:

$$H(u) = \frac{1}{2}u'^2 + F(u) = \text{constant} \tag{6}$$

From Eq. (6), we have;

$$\frac{\partial H}{\partial A} = 0 \tag{7}$$

Introducing a new function, $\bar{H}(u)$, defined as;

$$\bar{H}(u) = \int_0^{T/4}\left\{\frac{1}{2}u'^2 + F(u)\right\}dt = \frac{1}{4}TH \tag{8}$$

Eq. (7) is, then, equivalent to the following one;

$$\frac{\partial}{\partial A}\left(\frac{\partial \bar{H}}{\partial T}\right) = 0 \tag{9}$$

or

$$\frac{\partial}{\partial A}\left(\frac{\partial \bar{H}}{\partial (1/\omega)}\right) = 0 \tag{10}$$

From Eq. (10) we can obtain approximate frequency–amplitude relationship of a nonlinear oscillator.

Application

Solution using Hamiltonian approach

The Hamiltonian of Eq. (1) is constructed as:

$$H = \frac{1}{2}\left(\frac{du}{dt}\right)^2 + \frac{1}{2}\varepsilon_1\left(\frac{du}{dt}\right)^2 u^2 + \frac{1}{2}u^2 + \frac{1}{4}\varepsilon_2 u^4 \tag{11}$$

Integrating Eq. (11) with respect to t from 0 to $T/4$, we have:

$$\bar{H} = \int_0^{T/4}\left(\frac{1}{2}\left(\frac{du}{dt}\right)^2 + \frac{1}{2}\varepsilon_1\left(\frac{du}{dt}\right)^2 u^2 + \frac{1}{2}u^2 + \frac{1}{4}\varepsilon_2 u^4\right)dt \tag{12}$$

Assume that the solution can be expressed as:

$$u(t) = A\cos(\omega t) \tag{13}$$

Substituting Eq. (13) into Eq. (12), we obtain:

$$\begin{aligned}\bar{H} &= \int_0^{T/4}\left(\begin{array}{l}\frac{1}{2}\mathrm{A}^2\,\omega^2\sin^2(\omega \mathrm{t})+\frac{1}{2}\varepsilon_1\ \mathrm{A}^4\,\omega^2\sin^2(\omega \mathrm{t})\cos^2(\omega t)+\frac{1}{2}\mathrm{A}^2\cos^2(\omega t)+\\ +\frac{1}{4}\varepsilon_2\mathrm{A}^4\cos^4(\omega t)\end{array}\right)dt\\ &= \int_0^{\pi/2}\left(\frac{1}{2}\mathrm{A}^2\,\omega\sin^2 t+\frac{1}{2}\varepsilon_1\ \mathrm{A}^4\,\omega\sin^2 t\,\cos^2 t+\frac{1}{2\omega}\mathrm{A}^2\cos^2 t+\frac{1}{4\omega}\varepsilon_2\mathrm{A}^4\cos^4 t\right)dt\\ &= +\frac{1}{8}\omega A^2\pi+\frac{1}{32}\omega A^4\varepsilon_1\pi\frac{1}{8\omega}A^2\pi+\frac{3}{64\omega}A^4\varepsilon_2\pi\end{aligned} \tag{14}$$

Setting:

$$\frac{\partial}{\partial A}\left(\frac{\partial\bar{H}}{\partial(1/\omega)}\right) = -\frac{1}{4}A\,\pi\,\omega^2-\frac{1}{8}\varepsilon_1 A^3\,\pi\,\omega^2+\frac{1}{4}A\,\pi+\frac{3}{16}\varepsilon_2\,A^3\,\pi \tag{15}$$

Solving the above equation, an approximate frequency as a function of amplitude is equal to:

$$\omega = \frac{\sqrt{2}}{2}\frac{\sqrt{(\varepsilon_1 A^2+2)(3\varepsilon_2 A^2+4)}}{(\varepsilon_1 A^2+2)} \tag{16}$$

Hence, the approximate solution can be readily obtained:

$$u(t) = A\cos\left(\frac{\sqrt{2}}{2}\frac{\sqrt{(2+\varepsilon_1 A^2)(4+3\varepsilon_2 A^2)}}{(2+\varepsilon_1 A^2)}\ t\right) \tag{17}$$

Results and discussions

The exact frequency ω_{ex} for a dynamic system governed by Eq. (1) can be derived, as shown in Eq. (18), as follows:

$$\omega_{Exact} = 2\pi \Big/ 4\sqrt{2}\,A\int_0^{\pi/2}\frac{\sqrt{1+\varepsilon_1 A^2 cos^2 t}\ \sin t}{\sqrt{A^2(1-cos^2 t)(\varepsilon_2 A^2 cos^2 t+\varepsilon_2 A^2+2)}}\,dt \tag{18}$$

Some comparisons are presented to illustrate and verify the accuracy of the Hamiltonian approach. Table 1 presents the comparison of frequencies obtained with the HA and the exact ones for different value of A, ε_1 and ε_2. The maximum relative error between the HA results and exact results is 2.6846%. Figure 2 and 4 provide a comparison of analytical solution of $u(t)$ based on time with the numerical solution and figure 3 and 5 show comparison of analytical solution of du/dt based on time with the exact solution. From Figs.2-5, the motion of the system is a periodic motion and the amplitude of vibration is a function of the initial conditions. Comparison of frequency corresponding to various parameters of amplitude (A) and

ε_2 for $\varepsilon_1 = 0.5$ has been studied in the figure (Fig. 6). The effect of small parameters ε_1 on the frequency corresponding to various parameters of amplitude (A) has been studied in Fig. 3 for $\varepsilon_2 = 1$. It is evident that the Hamiltonian approach exhibits an excellent agreement with the exact solution and quickly convergent and is valid for a wide range of vibration amplitudes and initial conditions. The accuracy of the results shows that the Hamiltonian approach can be potentiality used for the accurate analysis of strongly nonlinear oscillation problems.

Table 1. Comparison of frequency corresponding to various parameters of system

Constant parameters			Approximate solution	Exact solution	Relative error %
A	ε_1	ε_2	ω_{HA}	ω_{Exact}	$\left\lvert \frac{\omega_{EX} - \omega_{HA}}{\omega_{Ex}} \right\rvert$
0.1	0.1	0.1	1.0001	1.0005	0.0374
0.1	1	0.2	0.9983	0.9983	0.0002
0.5	0.5	1	1.0572	1.0573	0.0084
0.5	1	0.5	0.9860	0.9870	0.1018
1	1	1	1.0801	1.0904	0.9382
1	0.5	0.2	0.9592	0.9623	0.3262
2	0.4	0.2	0.9428	0.9593	1.7212
2	1	0.8	1.0646	1.0917	2.4853
2	1	0.2	0.7303	0.7504	2.6846

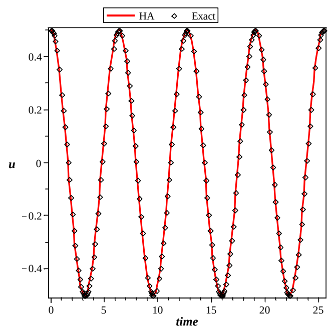

Fig. 2. Comparison of analytical solution of $u(t)$ based on time with the exact solution for $\varepsilon_1 = 1, \quad \varepsilon_2 = 0.5, \quad A = 0.5$

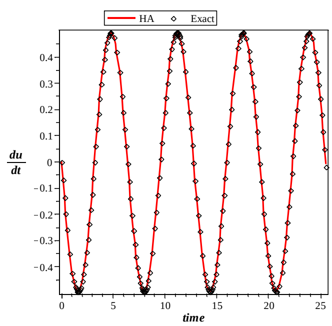

Fig. 3. Comparison of analytical solution of du/dt based on time with the exact solution for $\varepsilon_1 = 1, \; \varepsilon_2 = 0.5, \; A = 0.5$

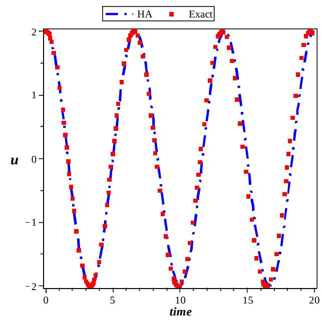

Fig. 4. Comparison of analytical solution of $u(t)$ based on time with the exact solution for $\varepsilon_1 = 0.4$, $\varepsilon_2 = 0.2$, $A = 2$

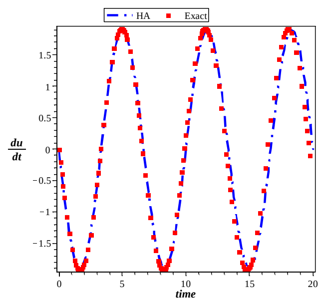

Fig. 5. Comparison of analytical solution of du/dt based on time with the exact solution for $\varepsilon_1 = 0.4$, $\varepsilon_2 = 0.2$, $A = 2$

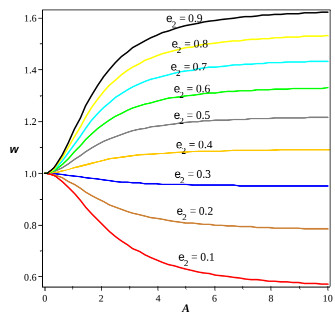

Fig. 6. Comparison of frequency corresponding to various parameters of amplitude (A) for $\varepsilon_1 = 0.5$

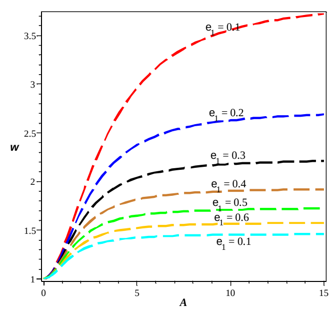

Fig. 7. Comparison of frequency corresponding to various parameters of amplitude (A) for $\varepsilon_2 = 1$

Conclusions

In this paper, a novel method has been used to obtain analytical solutions for the nonlinear vibration of a tapered beam. The analytical solutions yield a thoughtful and insightful understanding of the effect of system parameters and initial conditions. As shown in this study an excellent agreement between the approximate frequencies and the exact one are demonstrated and discussed. Hamiltonian approach can be a powerful mathematical tool for studying the strong nonlinear problems. Its excellent accuracy in the whole range of oscillation amplitude values is one of the most significant features of this method. Hamiltonian approach requires smaller computational effort and only a single iteration leads to accurate solutions. The successful implementation of the Hamiltonian approach for the large-amplitude nonlinear oscillation problem considered in this paper further confirms the capability of those methods in solving nonlinear oscillation problems. We can suggest this approach as a novel and simple method for oscillation systems, which provides an easy and direct procedure for determining approximations to the periodic solutions.

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