658. Frequency-based crack identification for static beam with rectangular cross-section

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Abstract. Aiming at the most dangerous crack damage in structural health, a crack identification method based on wavelet finite element model and determinant transformation method is conducted. Natural frequencies of the structure with various crack locations and depths are first accurately obtained by means of wavelet finite element methods. Then, the actual structures are measured to gain vibration characteristics. Measured natural frequencies are used in a crack detection process, while crack location and size can be identified using determinant transformation method. The experimental example of a static beam with rectangle cross-section indicates that the current method is effective and accurate. The whole approach provides a powerful technology for practical application.

Keywords: crack detection, vibration, wavelet finite element method, determinant transformation.

1. Introduction

Vibration-based methods have so far been intended for exploitation in structural crack detection [1-4]. In the presented approach, some signal features, such as change in natural frequencies, change in mode shapes, and change in amplitude of vibration have been taken into account. The natural frequency of a structure is most easily measured from accessible point on the static component and convenient to use. Also, such measurement method is fast, easy and inexpensive.

The frequency-based method includes two procedures [1]. The first procedure is forward problem, which comprises the construction of crack model exclusively for crack section and the construction of a numerical structural model to gain crack detection database for natural frequencies. That is the determination of function G_s relationship between the first three natural frequencies ω_s, crack normalized location β and crack normalized depth α, as follows:

$$\omega_s = G_s(\beta, \alpha) \qquad (s = 1, 2, 3) \tag{1}$$

The second procedure is referred to as inverse problem, which consists of the measurement of modal parameters and the detection of crack parameters. That is the determination of crack normalized location β and depth α as follows:

$$(\beta, \alpha) = G_s^{-1}(\omega_s) \qquad (s = 1, 2, 3) \tag{2}$$

The scheme of crack identification problem is depicted in Fig. 1.

In the forward problem studies, Nandwana [2] modeled the crack as a rotational spring and gave a semi-analytical solution for beams. Meanwhile, the finite element method (FEM) was employed for the identification of a crack in structures due to the fact that FEM is firmly established as a standard procedure for the solution of crack problems. Lele [3] employed finite elements to make a more efficient calculation for crack identification in a short beam with rectangular section. Because of the fact that the crack tip field displacement and stress have $1/\sqrt{\tau}$ singularity (τ denotes crack tip field radius in polar coordinates) and the traditional

FEM piecewise polynomial cannot approximate them accurately on a local area, a fine mesh and great amount of computational work is required when the traditional finite elements are used to describe the singular behavior of cracks. To overcome these difficulties, wavelets have been applied to finite element analysis because wavelet multiresolution theory provides a powerful mathematical tool for function approximation and multiscale representations. Unlike traditional FEM, Wavelet finite element method (WFEM) was utilized for model analysis of crack problems with good performance [5, 6]. According to linear fracture mechanics theory, the localized additional flexibility in crack vicinity can be represented by a lumped parameter element. The cracked beam is modeled by wavelet-based elements to gain crack detection database [5, 6].
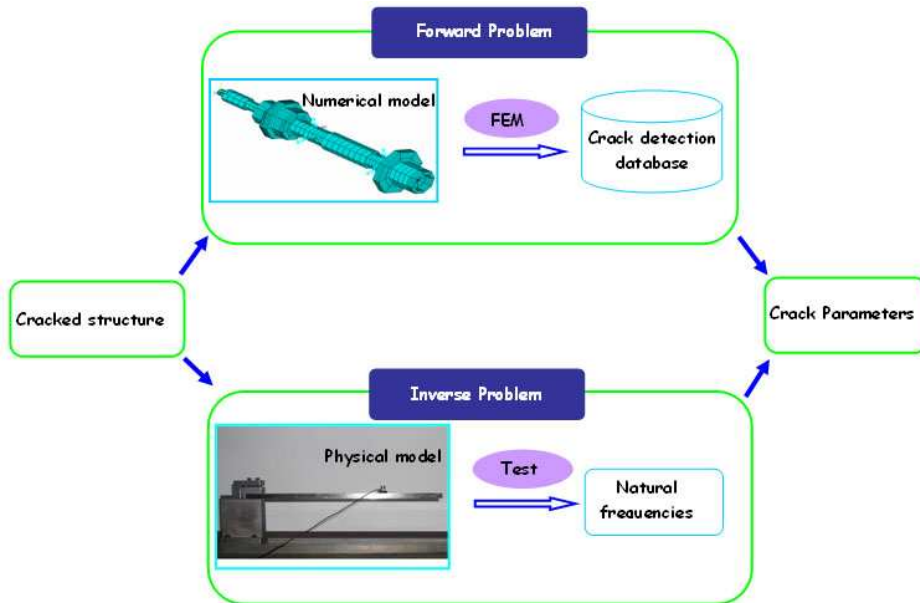

Fig. 1. Scheme for crack identification

With accurately measured frequencies after FFT, the solution of inverse problem for crack identification can be essentially an optimization problem. Several algorithms such as genetic algorithm [7], neural network [8], support vector machines [9], single-variable or multiobjective optimization algorithms [10, 11], fuzzy Gaussian inference technique [12], Bayesian parameter estimation [13], and frequency contour [14-16] were employed as optimization methods to minimize the errors between numerical simulation and experimental measurement.

Due to the facts that frequency contour method is visualized and easy utilized in practice, it became mostly popular algorithm in crack identification problems. However, the three-dimensional surfaces of the natural frequencies are based on macro-calculation in different crack locations and sizes, which influence the efficiency of the method in terms of contour lines.

In this paper, a crack identification algorithm based on wavelet finite element model and determinant transformation method is proposed. The natural frequencies of the structure with various crack locations and depths are first accurately obtained by means of WFEM. Then, the real structures are measured to gain vibration characteristics. Measured natural frequencies after FFT analysis are used in a crack detection process. Crack location and size can be identified using determinant transformation method. The experimental example of a static beam with rectangle cross-section indicates that the current method is effective and accurate. The whole approach provides a powerful technology for practical application.

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2. Crack identification method using WFEM and determinant transformation

2.1. Wavelet finite element method (WFEM)

WFEM is a new FEM which combines wavelet multiresolution analysis and conventional finite element variation principle. The good properties of Daubechies wavelet scaling functions (Fig. 2) and wavelet functions in locality and smooth enable WFEM to approximate finite element solution space with minimum basis. In addition, the sequence of closed subspaces guarantee WFEM to converge without saturation property supposition, which takes convenience for algorithm development of adaptive WFEM. For a one-dimensional wavelet-based finite element, the nodal displacements can be represented by the shape functions, whose forms are as follows:

$$N = \varphi T,$$ (3)

where T stands for the transformation matrix, and its elements are t_{ij} [15]. φ denotes the Daubechies wavelet [17] scaling function collection. After constructing wavelet-based shape functions, the forming procedures of the stiffness matrix K_e and the mass matrix M_e can be achieved as was done in the traditional FEM. The forms of both matrixes are represented as:

$$K_e = \int_{\Omega} (LN)^{\mathrm{T}} D (LN) \, d_x d_y,$$ (4)

$$M_e = \int_{\Omega} \rho \, N^{\mathrm{T}} N \, d_x d_y,$$ (5)

where L and D denote the generalized strain matrix and the elasticity matrix, respectively. ρ is the beam density. The superscript T stands for the matrix or vector transpose.

2.2. Modal analysis of a cracked beam

A uniform beam with an open crack is shown in Fig. 3. L, h, and b represent the length, height and width of the beam respectively. l and a are the crack location and crack size respectively. β ($\beta = l/L$) and α ($\alpha = a/h$) stand for the normalized crack position and normalized crack size respectively.

Suppose that the crack is located between two wavelet finite elements, and the numbers of two nodes are i and $i+1$ respectively (See Fig. 4).

The crack introduces a local flexibility that is a function of the crack depth, and the flexibility changes the stiffness of the beam. Rizos, Aspragathos, and Dimarogonas [18] represented the crack by a mass-less rotational spring with a computable stiffness Kt. The values of Kt for various cross-sections were given by Dimarogonas [19]. The continuity conditions at the crack position indicate that the left node and right node have the same vertical deflection, $y_i = y_{i+1}$, while their rotations θ_i and θ_{i+1} are connected through the stiffness matrix Kc [20]:

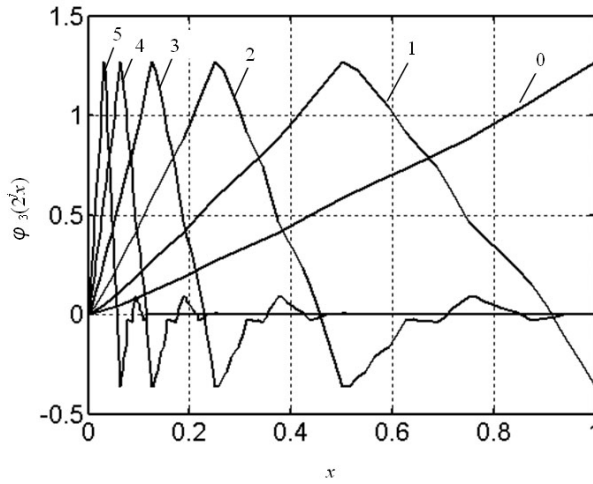$$K_c = \begin{bmatrix} Kt & -Kt \\ -Kt & Kt \end{bmatrix}.$$ (6)

Fig. 2. Daubechies wavelet scaling function of D3
(0: *j*=0, 1: *j*=1, 2: *j*=2, 3: *j*=3, 4: *j*=4, 5: *j*=5)
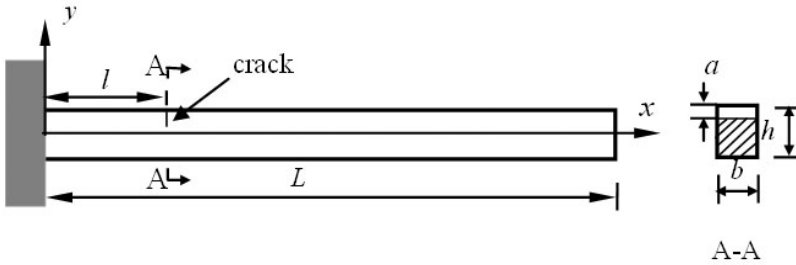

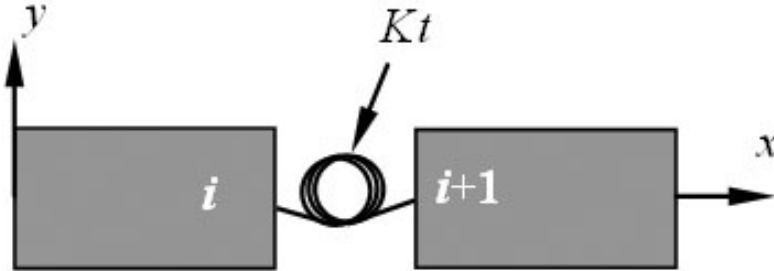**Fig. 3.** The model of cantilever beam with an open crack

Fig. 4. Layout of the corresponding nodes around crack

Hence, ***Kc*** can be assembled into the global stiffness matrix of WFEM through employing a single degree-of-freedom of the vertical deflection of both nodes *i* and *i*+1. The global mass matrix of the cracked beam is equal to the uncracked one.

For the cantilever beam shown in Fig. 3, the boundary conditions caused by fixed support are as follows:

$$w(x) = 0 , \quad \dot{w}(x) = 0 , \quad x = 0 \quad \text{or} \quad x = L \tag{7}$$

where w is a lateral displacement, overdots indicate differentiation with respect to time.

The boundary conditions caused by the crack are as follows:

$$
\begin{cases}
w_i(x) = w_{i+1}(x) \\[2ex]
\dfrac{d^2 w_i(x)}{d\beta^2} = \dfrac{d^2 w_{i+1}(x)}{d\beta^2} \\[2ex]
\dfrac{d^3 w_i(x)}{d\beta^3} = \dfrac{d^3 w_{i+1}(x)}{d\beta^3} \\[2ex]
\dfrac{dw_i(x)}{d\beta} + \dfrac{EI}{KtL}\dfrac{d^2 w_i(x)}{d\beta^2} - \dfrac{dw_{i+1}(x)}{d\beta} = 0
\end{cases}
\tag{8}
$$

where $w_i(x)$ is lateral displacement on left side node of the crack, $x \in [0, l]$. $w_{i+1}(x)$ is lateral displacement on right side node of the crack, $x \in [0, l]$. E is Young's modulus, and I is inertia moment.

Using the displacement-based formulation in conjunction with the principle of virtual displacement, the equations of the beam motion can be written as:

$$M\ddot{Y} + KY = 0, \tag{9}$$

where overdots indicate differentiation with respect to time, M an K are the global mass and stiffness matrix respectively. Y ($= [y_1, \theta_1, \ldots, y_n, \theta_n]$) is the column vector of nodal displacements.

Suppose that a time harmonic solution for the nodal displacements can be represented as:

$$Y = A \sin \omega_s t, \tag{10}$$

where A is the amplitude of the nodal displacements, and ω_s is the natural frequency. Substituting Eq. (10) into Eq. (9) leads to:

$$[-\omega_s^2 M + K]\{A\} = 0. \tag{11}$$

For non-trivial solutions:

$$\det(K - \omega_s^2 M) = 0, \tag{12}$$

where "det" denotes the determinant. Therefore the effective values of the natural frequencies can be found by solving generalized eigenvalues of Eq. (12).

2.3. Determinant transformation

In section B, the finite element model of cracked beam is modeled by using WFEM. The crack is represented as a rotational spring. Utilizing the determinant transformation method it transforms the vibration frequency equation into the quadratic equation with one unknown parameter: the rotational spring stiffness. Finding the roots of quadratic equations at different crack locations, the three curves of spring stiffness versus crack location are plotted. The point of intersection of the curves identifies the location and size of the crack.

Suppose that the crack is located between two wavelet-based finite elements, and the numbers of two nodes are i and $i+1$, respectively. The spring stiffness Kt was an unknown parameter of the vibration frequency equation, Eq. (12):

$$|\boldsymbol{\Theta}| = 0 \tag{13}$$

or:

$$\begin{vmatrix} k_{1,1} - \lambda_m m_{1,1} & \cdots & \cdots & \cdots & \cdots & k_{1,n} - \lambda_m m_{1,n} \\ & \vdots & \vdots & \vdots & \vdots & \vdots \\ & \vdots & k_{i,i} - \lambda_m m_{i,i} + Kt & k_{i,i+1} - \lambda_m m_{i,i+1} - Kt & \vdots & \vdots \\ & \vdots & k_{i+1,i} - \lambda_m m_{i+1,i} - Kt & k_{i+1,i+1} - \lambda_m m_{i+1,i+1} + Kt & \vdots & \vdots \\ & \vdots & \vdots & \vdots & \vdots & \vdots \\ k_{n,1} - \lambda_m m_{n,1} & \cdots & \cdots & \cdots & \cdots & k_{n,n} - \lambda_m m_{n,n} \end{vmatrix} = 0 \tag{14}$$

where, $\lambda_s = \omega_s^2$, $(s = 1, 2, 3)$ are known.

According to the determinant calculation properties, the left determinant of Eq. (14) was expanded by ith column and $i+1$th column, and the quadratic equation with one unknown number could be obtained:

$$a(1)Kt^2 + a(2)Kt + a(3) = 0 \tag{15}$$

where:

$$a(1) = \begin{vmatrix} & 0 & \\ \boldsymbol{\Theta}(1:n,1:i-1) & X & \boldsymbol{\Theta}(1:n,i+2:n) \\ & 0 & \end{vmatrix} \tag{16}$$

$$a(2) = \begin{vmatrix} & 0 & \\ \boldsymbol{\Theta}(1:n,1:i-1) & H & \boldsymbol{\Theta}(1:n,i+1:n) \\ & 0 & \end{vmatrix} + \begin{vmatrix} & 0 & \\ \boldsymbol{\Theta}(1:n,1:i) & N & \boldsymbol{\Theta}(1:n,i+2:n) \\ & 0 & \end{vmatrix} \tag{17}$$

$$a(3) = |\boldsymbol{\Theta}| \tag{18}$$

$\boldsymbol{\Theta}(i:j,k:l)$ is the sub-matrix formed by the elements (from the ith row to jth row, and the kth column to lth column of $\boldsymbol{\Theta}$). $X = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$, $H = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $N = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

The natural frequencies were taken into the Eq. (15), the corresponding spring stiffness Kt was obtained by finding the roots of Eq. (15). The same calculation was repeated in a different crack location β, so we can get three curves of $\beta - Kt$. The crossing points present the crack location β (horizontal ordinate) and spring stiffness Kt (longitudinal coordinate).

Because the crack stiffness Kt could be expressed as:

$$Kt = \frac{bh^2 E}{72\pi (a/h)^2 f(a/h)} \tag{19}$$

where:

$$f(a/h) = 0.6384 - 1.035(a/h) + 3.7201(a/h)^2 - 5.1773(a/h)^3 \\ + 7.553(a/h)^4 - 7.332(a/h)^5 + 2.4909(a/h)^6, \tag{20}$$

taking Kt into the Eq.(19), we can get the crack depth a.

2. Experimental example

In order to verify the usefulness of the developed method, the experimental equipment was built (Fig. 5). The cantilever beam with two crack cases is divided into 8 wavelet finite elements, D6 [15] during analysis. The geometric and material properties of the beam are: $L = 0.5\,\mathrm{m}$, $h = 0.02\,\mathrm{m}$, $b = 0.012\,\mathrm{m}$, $E = 2.1\,\mathrm{GPa}$, $\rho = 7860\,\mathrm{Kg \cdot m^{-3}}$ and $\upsilon = 0.3$. Fig. 6 provides the relationship between ω_s, $(s = 1,2,3)$ and possible normalized crack location using WFEM.

(a)

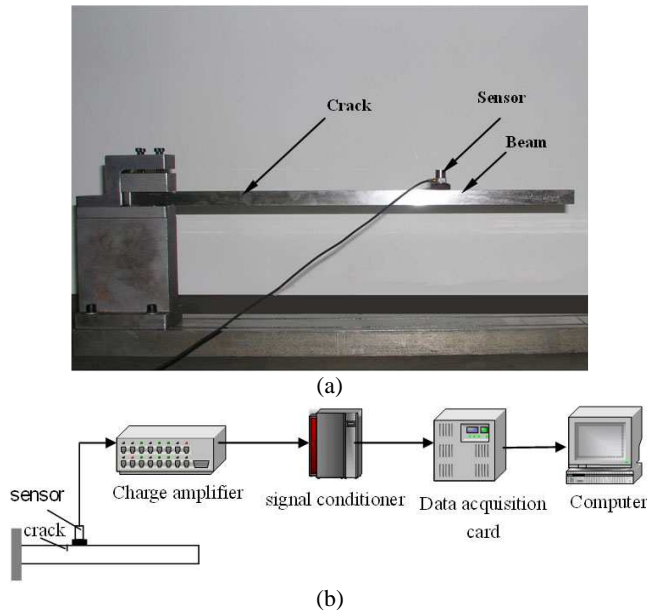(b)

Fig. 5. Experimental equipment: (a) Experimental setup; (b) Chart of measuring principle
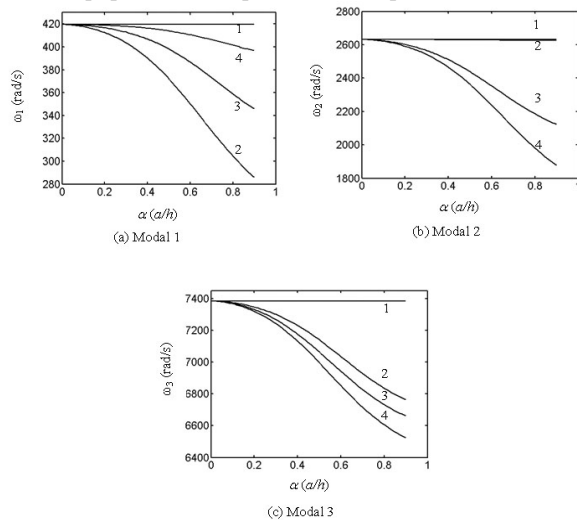

(a) Modal 1

(b) Modal 2

(c) Modal 3

Fig. 6. Relationship between natural frequencies and crack location

In order to select the best testing location of a sensor, two facts are considered. Fig. 7 illustrates the first three vibration shapes of the cantilever beam. The free end of the beam has the highest value of lateral displacement and acceleration, so the point near to the free end of beam is most sensitive to the vibration. On the other side, the same point is also the best place to be excited by hamper. Based on our experience, the best point of locating sensor is the place where x/L is 0.38.
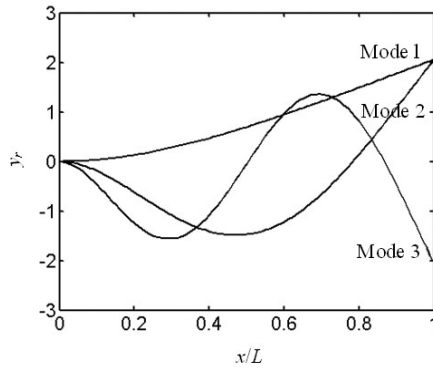

Fig. 7. The first three vibration shapes of a cantilever beam

The measured natural frequencies ω^*_s, $(s=1,2,3)$ are used as the input parameters in order to produce the predicted crack variables (β, α). The intersection of the three curves in Fig. 8 indicates possible crack position and crack size. When the three curves do not meet exactly, the midpoint of the three pairs of intersections is taken as the crack position and crack size. The estimations of crack variables are given by Table 1. The results based on WFEM and determinant transformation indicate that the normalized error of crack location is less than 2%, while the error of crack depth is less than 13%.
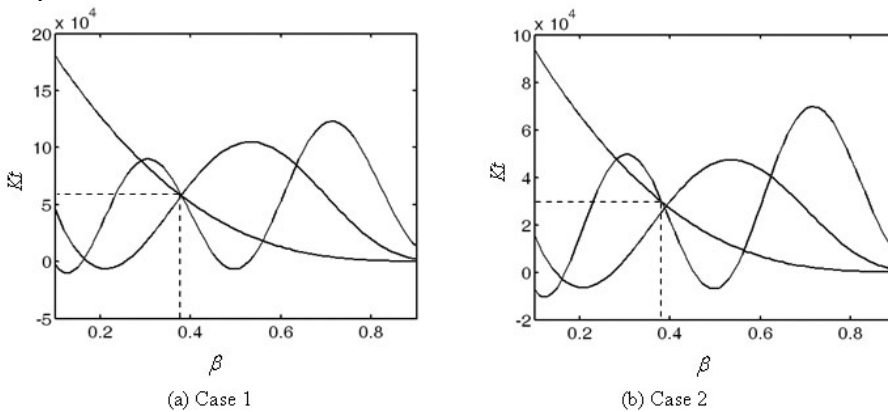

Fig. 8. Predicted crack variables for the two crack cases

Table 1. Predicted crack variables for the two crack cases

No.	Crack location β	Crack size a/h	β^*	a^*/h	β error %	a/h error %
1	0.4	0.2	0.38	0.33	2.0	13.0
2	0.4	0.4	0.39	0.48	1.0	8.0

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3. Conclusions

Utilizing the properties of determinant transformation, this paper deduced the quadratic equation with one unknown parameter: the rotational spring stiffness from the vibration frequency equation. The inverse problem of crack identification can be represented as solving for the roots of quadratic equations at different crack locations. Experimental investigations verify that the proposed method can be utilized to detect crack location as well as crack size with high accuracy. This study provides a new method for the prognosis and diagnosis of cracks in various structures.

Acknowledgements

References

[1] **B. Li, X. F. Chen, Z. J. He** "Identifying a Crack in a Cantilever Beam based on the Determinant Transformation Method", *Proceedings of the 23 nd International Congress on Condition Monitoring and Diagnostic Engineering Management COMADEM 2010*, Japan, p. 393-399, 2010.

[2] **B. P. Nandwana, S. K. Maiti** "Detection of the location and size of a crack in stepped cantilever beams based on measurements of natural frequencies", *J. Sound Vib.* Vol. 203, p. 435–446, 1997.

[3] **S. P. Lele, S. K. Maiti** "Modeling of transverse vibration of short beams for crack detection and measurement of crack extension", *J. Sound Vib.* Vol. 257, p. 559–583, 2002.

[4] **M. R. Naniwadekar, S. S. Naik, S. K. Maiti** "On prediction of crack in different orientations in pipe using frequency based approach". *Mech. Syst. S.*, Vol. 22, p. 693–708, 2008.

[5] **H. B. Dong, X. F. Chen, B. Li, K. Y. Qi, Z. J. He** "Rotor crack detection based on high-precision modal parameter identification method and wavelet finite element model", *Mech. Syst. S.*, Vol. 23, p. 869-883, 2009.

[6] **J. W. Xiang, X. F. Chen, B. Li., Y. M. He, Z. J. He** "Identification of a crack in a beam based on the finite element method of a B-spline wavelet on the interval", *J. Sound Vib.* Vol. 10, p. 1046-1052, 2006.

[7] **M. T. Vakil-Baghmisheh, M. Peimani** "Crack detection in beam-like structures using genetic algorithms", *Applied Soft Computing*, Vol. 8, p. 1150-1160, 2008.

[8] **B. S. Wang, Z. C. He** "Crack detection of arch dam using statistical neural network based on the reductions of natural frequencies", *J. Sound Vib.* Vol. 302, p. 1037-1047, 2007.

[9] **Y. T. Zhong, J. W. Xiang, Z. S. Jiang** "Pipe damage detection method by combination of wavelet-based element and support vector regression", *J. of Vibroengineering*, Vol. 12, p. 269-277, 2010.

[10] **H. Ressing, M. S. Gadala** "A practical investigation to solving the inverse problem of crack identification through vibration measurements", *Emerald*. Vol. 23, p. 32-56, 2006.

[11] **R. Perera, S. E. Fang, C. Huerta** "Structural crack detection without updated baseline model by single and multiobjective optimization". *Mech. Syst. S.*, Vol. 23. p. 752-768, 2009.

[12] **H. C. Das, D. R. Parhi** "Detection of the Crack in Cantilever Structures Using Fuzzy Gaussian Inference Technique". *AIAA J.*, Vol. 47, p. 105-115, 2009.

[13] **E. Z. Moore, K. D. Murphy, J. M. Nichols** "Crack identification in al freely vibrating plate using Bayesian parameter estimation", *Mech. Syst. S.*, 2011, in press.

[14] **G. M. Owolabi, A. S. J. Swamidas, R. Seshadri** "Crack detection in beams using changes in frequencies and amplitudes of frequency response functions", *J. Sound Vib.*, Vol. 265 p. 1-22, 2003.

[15] **B. Li, X. F. Chen, Z. J. He** "Detection of Crack Location and Size in Structures Using Wavelet Finite Element Methods", *J. Sound Vib.*, Vol. 285, p. 767-782, 2005.

[16] **H. Nahvi, M. Jabbari** "Crack detection in beams using experimental modal data and finite element model", *Int. J. Mech.*, Vol. 47, p. 1477-1497, 2005.

[17] **I. Daubechies** "Orthonormal bases of compactly supported wavelets", *Com. Pa. Math.*, Vol. 41, p. 909-996, 1998.

[18] **P. F. Rizos, N. Aspragathos, A. D. Dimarogonas** "Identification of crack location and magnitude in a cantilever beam from the vibration modes", *J. Sound Vib.,* Vol. 138, p. 381-388, 1990.
[19] **A. D. Dimarogonas** "Vibration of cracked structures: a state of the art review. Engineering", *Eng. Fract. M.*, Vol. 55, p. 831-857, 1996.
[20] **S. Chinchalkar** "Determination of crack location in beams using natural frequencies", *J. Sound Vib.,* Vol. 247, p. 417-429, 2001.