652. Effect of physical nonlinearity on bending vibrations of elements of packages

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Abstract. Bending vibrations of elements of packages are analyzed numerically by using a beam-type model containing physical nonlinearity. The model for the analysis of skeletal curves of bending vibrations of package element is presented. The 1D model with physical nonlinearity is based on cubic nonlinearity of Duffing type. Phase trajectory of steady state motion of the eigenmode is obtained and analyzed. The model for the analysis of bending vibrations of statically loaded physically nonlinear element of package is presented. First the static problem is solved and than the eigenproblem of small vibrations about the statically deflected structure is analyzed by taking into account cubic nonlinearity of Duffing type. It is demonstrated that the eigenmodes and eigenvalues are influenced by the physical nonlinearity. The setup for experimental investigation of polymeric films for symmetrically distributed loading is presented. The method of projection moiré is applied for this purpose. In the process of investigations the images of the first four eigenmodes of the polymeric HDPE film were determined. The obtained results are used for designing elements of packages.

Keywords: physical nonlinearity, nonlinear elasticity, finite elements, cubic nonlinearity, Duffing equation, vibrations, eigenmodes, skeletal curves, phase trajectory, steady state motion, Duffing parameter, eigenvalues, non-destructive identification, time averaging, projection moiré.

Introduction

Bending vibrations of package elements are numerically analyzed by using beam-type model with physical nonlinearity. The model for the analysis of skeletal curves of bending vibrations of the element is presented. The one dimensional model with physical nonlinearity in the form of cubic nonlinearity of Duffing type is employed. The analysis is performed on the basis of the models used for studying beam bending as well as implementing modal decomposition of motion as described in [1-3]. It is established that the bending behavior is substantially influenced by the physical nonlinearity. Phase trajectory of steady state motion of the eigenmode is obtained and analyzed.

The model for the analysis of bending vibrations of statically loaded physically nonlinear package element is presented. Firstly, the static problem is solved and then the eigenproblem of small vibrations about the statically deflected structure is considered by taking in to account cubic nonlinearity of Duffing type. The investigation is performed on the basis of models described in [2, 3]. It is demonstrated that the eigenmodes and eigenvalues are influenced by the physical nonlinearity.

Some of the results of experimental investigation of eigenmodes of statically loaded paper are presented in [4-6]. The experiments revealed that when the static load is not very large it is much easier to estimate its effect by means of change of eigenfrequency rather than from the change of the mode shape. This corresponds to the conclusion made in this paper on the basis of numerical investigations for the physically nonlinear element of package using the model of a beam.

The method of projection moiré [7] is used for the experimental investigation of defects of polymeric materials as well as for performing their non-destructive diagnostics [8, 9]. The authors of this article studied the mechanical characteristics of the paper and paperboard by using the method of projection moiré under symmetric [10] and un-symmetric [11] loading of the tape of paper and paperboard.

The model for the analysis of physically nonlinear elastic structure was developed by the authors [12]. Nonlinear elasticity was taken into account using the hyperbolic model and considered as an approximation to the plastic behavior, which has been analyzed by the theory of deformational plasticity valid for monotonic loading. It was shown that the bending behavior is substantially influenced by the physical nonlinearity.

The obtained research results are applied in the process of design of packaging elements.

Model for the analysis of skeletal curves of nonlinear bending of elements of packages

Further x, y and z denote the axes of the system of coordinates. The bending element of package has two nodal degrees of freedom: the displacement w in the direction of the z axis and the rotation Θ_y about the y axis. The displacement u in the direction of the x axis is expressed as $u=z\Theta_y$.

Longitudinal strain is expressed as:

$$\varepsilon_x = z[B]\{\delta\}, \tag{1}$$

where:

$$[B]=\begin{bmatrix} 0 & \frac{dN_1}{dx} & \dots \end{bmatrix}, \tag{2}$$

where N_i are the shape functions of the finite element and $\{\delta\}$ is the vector of generalized displacements.

The following notation is introduced:

$$\bar{\varepsilon} = [B]\{\delta\}. \tag{3}$$

Physical nonlinearity is assumed in the expression of the longitudinal stress:

$$\sigma_x = \frac{E}{1-\nu^2}\left(\varepsilon_x + b\varepsilon_x^3\right), \tag{4}$$

where E is the modulus of elasticity, ν is the Poisson's ratio and b is the Duffing parameter.

From the previous equations it is obtained:

$$\sigma_x = \frac{E}{1-\nu^2} z[B]\{\delta\} + \frac{E}{1-\nu^2} z^3 b\bar{\varepsilon}^3. \tag{5}$$

Thus the following quantity for the eigenmode i is calculated:

$$a_i = \int \frac{E}{1-\nu^2} b\bar{\varepsilon}_i^4 \frac{h^5}{80} dx, \tag{6}$$

where $\bar{\varepsilon}_i$ denotes $\bar{\varepsilon}$ calculated for the eigenmode i, h is the thickness of the element of package and the following integral has been taken into account:

$$\int_{-\frac{h}{2}}^{\frac{h}{2}} z^4 dz = \frac{h^5}{80}. \tag{7}$$

So the modal equation for free vibrations according to a single eigenmode takes the form:

$$\frac{d^2 z_i}{dt^2} + \left(\lambda_i + a_i z_i^2\right) z_i = 0, \tag{8}$$

where z_i is the coefficient of the eigenmode *i*, *t* is the time variable and λ_i is the eigenvalue *i*.

For forced vibrations the modal equation has the form:

$$\frac{d^2 z_i}{dt^2} + \left(\lambda_i + a_i z_i^2\right) z_i = \{\delta_i\}^T \{F\}, \tag{9}$$

where $\{\delta_i\}$ is the *i*-th eigenmode and $\{F\}$ is the loading vector.

Further it is assumed that:

$$\{F\} = \{\tilde{F}\} \sin \omega t, \tag{10}$$

where $\{\tilde{F}\}$ is a constant vector and ω is the frequency of excitation. Then:

$$\frac{d^2 z_i}{dt^2} + \left(\lambda_i + a_i z_i^2\right) z_i = A_i \sin \omega t, \tag{11}$$

where:

$$A_i = \{\delta_i\}^T \{\tilde{F}\}. \tag{12}$$

When damping is taken into account the modal equation takes the form:

$$\frac{d^2 z_i}{dt^2} + (\alpha + \beta\lambda_i)\frac{dz_i}{dt} + \left(\lambda_i + a_i z_i^2\right) z_i = A_i \sin \omega t, \tag{13}$$

where α is the coefficient of external damping and β is the coefficient of internal damping.

Results of calculation of skeletal curves of package elements

At both ends of the analyzed element of package both generalized displacements are assumed equal to zero. The eigenmodes for a linear problem are calculated. Then the coefficient of the nonlinear member of the modal equation for free vibrations is determined. This produces the Duffing equation the skeletal curve for which is calculated.

Skeletal curves for the first eight eigenmodes are presented in Fig. 1 (on the *x* axis the frequency divided by the first eigenfrequency $\sqrt{\lambda_1}$ is shown).

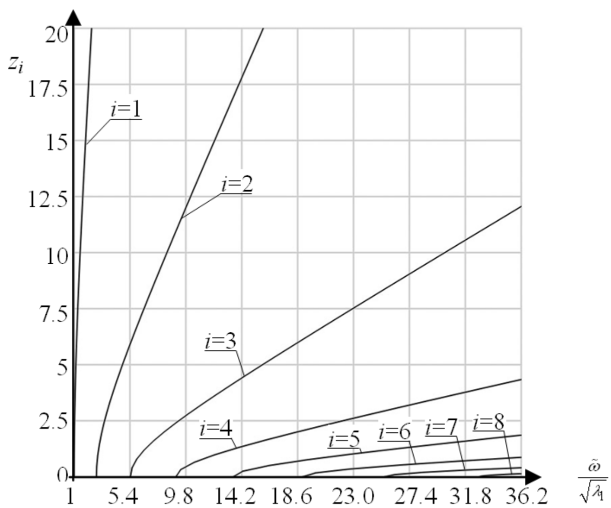


Fig. 1. Skeletal curves for the first eight eigenmodes

Phase trajectory of motion for the first eigenmode from the initial condition given by the prescribed value of $z_1(0)$ and with zero initial velocity for the system without excitation when $\alpha = 2\sqrt{\lambda_1}0.01$, $\beta = 0$ is presented in Fig. 2. Amplitude frequency characteristic of this motion is presented in Fig. 3 (on the *x* axis the frequency divided by the first eigenfrequency $\sqrt{\lambda_1}$ is shown).

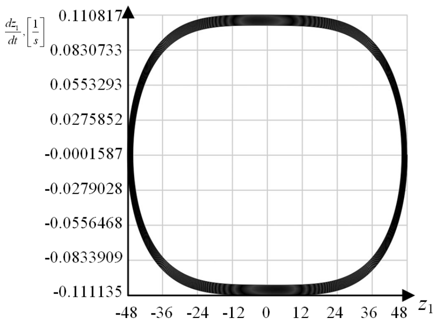


Fig. 2. Phase trajectory of the first eigenmode

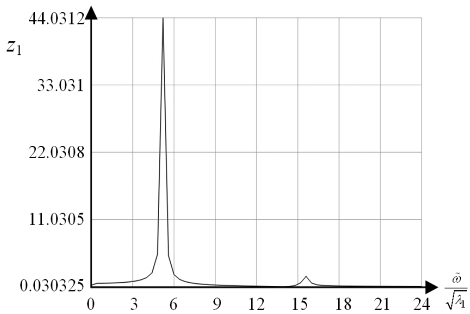


Fig. 3. Amplitude frequency characteristic

Phase trajectory of steady state motion of the first eigenmode when $\omega = 1.04\sqrt{\lambda_1}$, $\alpha = 2\sqrt{\lambda_1}0.02$, $\beta = 0$ is presented in Fig. 4. Amplitude frequency characteristic of this motion is presented in Fig. 5 (on the *x* axis the frequency divided by the frequency of excitation is shown).

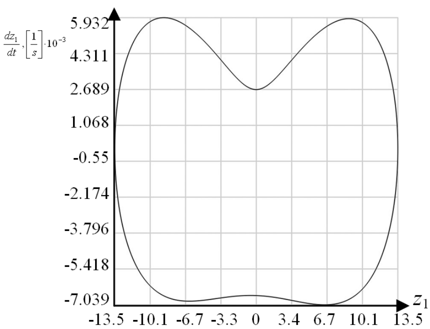


Fig. 4. Phase trajectory of the first eigenmode

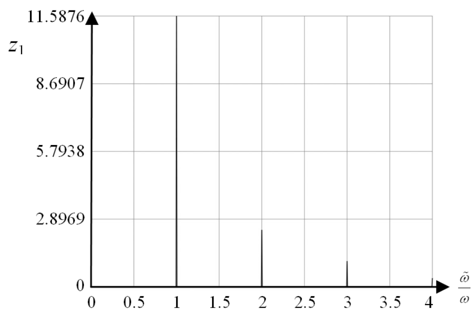


Fig. 5. Amplitude frequency characteristic

Amplitudes for slowly increasing frequency and for slowly decreasing frequency are illustrated in Fig. 6. The hysteresis loop for the first two eigenmodes is obtained and is evidently observed in the figure.

Model for the analysis of vibrations about a statically deflected element of package

The mass matrix has the form:

$$[M] = \int [N]^T \begin{bmatrix} \rho h & 0 \\ 0 & \rho \frac{h^3}{12} \end{bmatrix} [N] dx, \qquad (14)$$

where ρ is the density of the material of the element of package and:

$$[N]=\begin{bmatrix} N_1 & 0 & ... \\ 0 & N_1 & ... \end{bmatrix}. \tag{15}$$

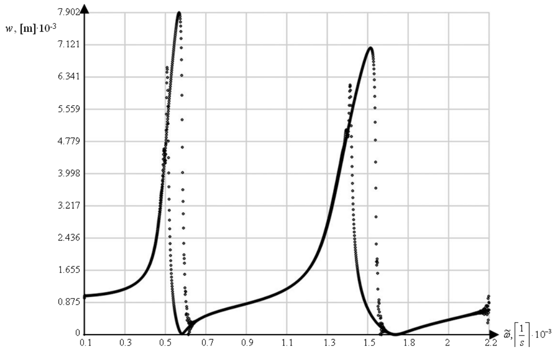


Fig. 6. Amplitudes for increasing frequency and for decreasing frequency

From the equations presented earlier it is obtained:

$$d\sigma_x=\frac{E}{1-\nu^2}\left(z+z^3 b3\bar{\varepsilon}^2\right)[B]\{d\delta\}. \tag{16}$$

The stiffness matrix has the form:

$$[K]=\int\left([B]^T\left[\frac{E}{1-\nu^2}\left(\frac{h^3}{12}+b3\bar{\varepsilon}^2\frac{h^5}{80}\right)\right][B]+\left[\bar{B}\right]^T\left[\frac{E}{2(1+\nu)1.2}h\right]\left[\bar{B}\right]\right)dx, \tag{17}$$

where:

$$\left[\bar{B}\right]=\begin{bmatrix}\frac{dN_1}{dx} & N_1 & ...\end{bmatrix}. \tag{18}$$

Results of analysis of vibrations about a statically deflected element of package

At both ends of the analyzed element of package both generalized displacements are assumed equal to zero, except for the deflection at the right end which is assumed equal to one. The following values of parameters are assumed: E = 8 N/m^2, ν = 0.3, h = 0.1 m, ρ = 0.8 kg/m^3, $b = 10^8$.

Graphical representation of the eigenmodes is given in Fig. 7. For a linear structure the eigenmodes are grey, while for a nonlinear one they are black.

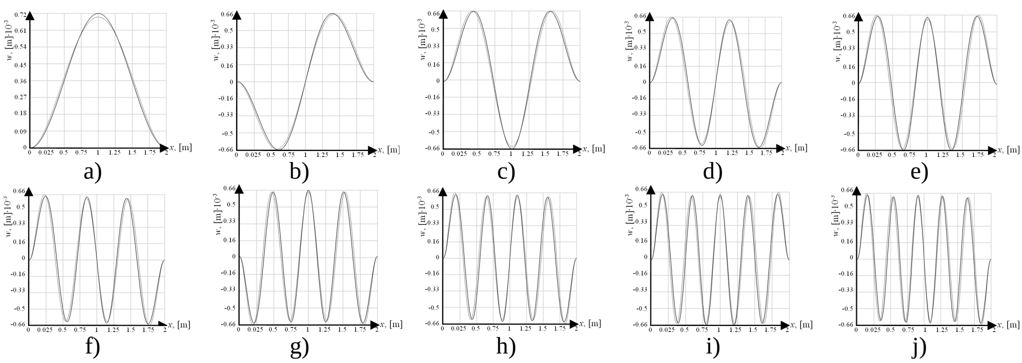
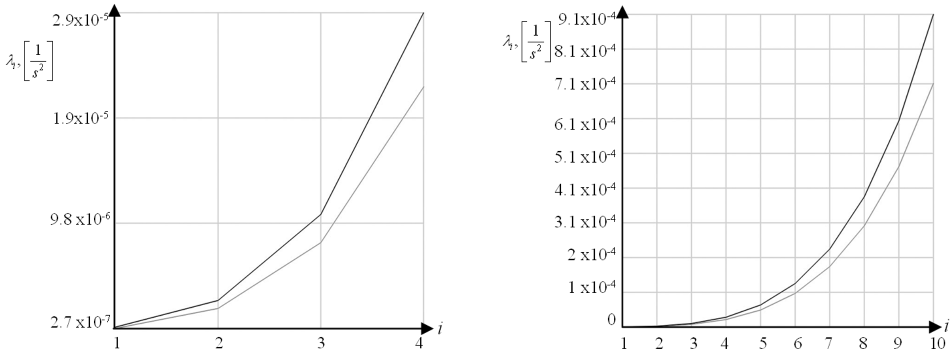


Fig. 7. The eigenmodes (for a linear structure the eigenmodes are grey, for a nonlinear one they are black): a) the first eigenmode, b) the second eigenmode, … , j) the tenth eigenmode

Graphical representation of the eigenvalues is given in Fig. 8. For a linear structure the eigenvalues are connected by a grey line, while for a nonlinear one they are connected by a black line.



a) b)

Fig. 8. The eigenvalues (for a linear structure the eigenvalues are connected by a grey line, for a nonlinear one they are connected by a black line): a) the first four eigenvalues, b) the first ten eigenvalues

The presented results demonstrate that the physical nonlinearity changes the eigenvalues more substantially than the eigenmodes.

Non-destructive identification of dynamical characteristics in elements of packages

In order to determine the dynamical characteristics of the elements of packages a special setup for experimental investigation was used [4-6, 9]. In the experiment the method of time-averaged projection moiré was implemented by projecting thin parallel lines of high contrast with the light flux incident to the surface of a vibrating polymeric film.

For the experimental investigations the polymeric HDPE film was chosen. Technical characteristics of this film are presented in Table 1.

Table 1. Technical characteristics of HDPE film

Surface density, g/m^2	61
Thickness, μm	60
Wetting angle Θ, °	95
Surface morphology R_a, μm	0.18
Gloss (45°), %	4.3

The sample of polymeric HDPE film with nonlinear structure obtained during the production of the polymeric film was used in the investigations. The experiments were performed in the longitudinal direction of production of the polymeric film. The ends of the investigated polymeric film were fastened between pressing tapes. One of the pressing tapes was fastened to the exciter of vibrations, while the other pressing tape was loaded symmetrically by the force of 25.5 N. Exciter of vibrations was generating longitudinal vibrations of sinusoidal shape of chosen frequency. Those vibrations excited standing waves in the analyzed material. Then by the method of time-averaged projection moiré the grid of thin parallel lines of high contrast was projected at a definite angle to the surface of the investigated material, and the first eigenmodes were obtained. The eigenmodes were photographed by using a digital camera and then they were processed in the personal computer.

The obtained results of the experimental investigations under symmetric tension of the polymeric film are presented in Fig. 9. As can be observed from Fig. 9, excitation of the polymeric HDPE film with periodic longitudinal vibrations of sinusoidal shape allows to produce the images of eigenmodes by means projection moiré that are sharp and clear with definite contours.

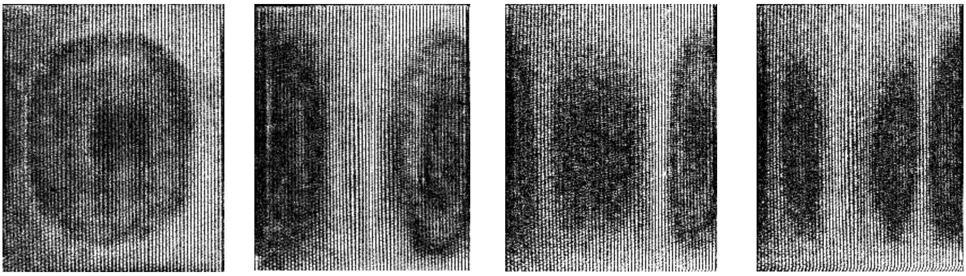


Fig. 9. The eigenmodes of the polymeric film HDPE: a) first, frequency of vibrations 76 Hz, amplitude 2×10^{-5} m; b) second, frequency of vibrations 86 Hz, amplitude 3.6×10^{-5} m; c) third, frequency of vibrations 105 Hz, amplitude 3.2×10^{-5} m; d) fourth, frequency of vibrations 109 Hz, amplitude 2.5×10^{-5} m

One is to have in mind the frequency range of vibrations and their amplitude range. When analyzing the changes of frequency and amplitude ranges in the polymeric film HDPE with the change of the number of the eigenmode one can note that for higher numbers of eigenmodes the frequency range of vibrations increases and the amplitude range becomes smaller (compare Fig. 9 and Table 2: the frequency range of vibrations increases from 71÷78 Hz up to 83÷94 Hz, and the amplitude range decreases from $(1.4\div5.8)\times10^{-5}$ m to $(2.2\div4.4)\times10^{-5}$ m).

Table 2. Generalized images of eigenmodes of HDPE polymeric film

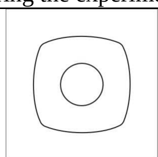
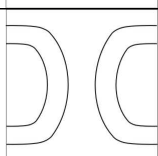
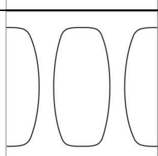
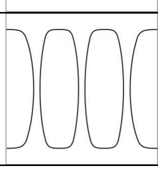
Eigenmode number	Geometric shape of the eigenmode, obtained during the experiments	Frequency range (Hz)	Amplitude range (m)
I		71 – 78 Hz	1.4 – 5.8 x 10^{-5} m
II		83 – 94 Hz	2.2 – 4.4 x 10^{-5} m
III		96 – 105 Hz	2.2 – 4.9 x 10^{-5} m
IV		107 – 116 Hz	1.8 – 3.1 x 10^{-5} m

Table 2 presents a summary of the study of the polymeric HDPE film by using projection moiré technique when the polymeric film was loaded symmetrically.

By comparing the moiré clouds presented in Fig. 9 and Fig. 7 one can note that the images of the eigenmodes obtained by using the method of projection moiré have exactly the same shape as the eigenmodes obtained by numerical investigations. The shapes of the first eigenmodes obtained by using the method of projection moiré consist of one, two, three or four moiré clouds (Fig. 9 a-d), and one can also count one – four peaks in the first eigenmodes obtained by using the numerical investigations (Fig. 7 a-d).

The fact that physical nonlinearity has small influence onto the shape of the eigenmode and more substantial influence onto the change of the eigenfrequency is confirmed both by the experimental and numerical investigations.

Conclusions

The model for the analysis of skeletal curves of bending vibrations of the elements of packages is presented. The one dimensional model with physical nonlinearity in the form of cubic nonlinearity of Duffing type is used. The eigenmodes are calculated. Then the coefficient of the nonlinear member of the modal equation for free vibrations is determined. This produces the Duffing equation the skeletal curve for which is calculated. It is demonstrated that the bending behavior is substantially influenced by the physical nonlinearity. Phase trajectory of steady state motion of the eigenmode is obtained and its amplitude frequency characteristic is determined.

Bending vibrations of statically loaded physically nonlinear element of package are investigated. First the static problem is solved and then the eigenproblem of small vibrations about the statically deflected structure is analyzed by taking into account cubic nonlinearity of Duffing type. From the presented results it is observed that physical nonlinearity changes the eigenvalues more substantially than the eigenmodes. Thus in order to identify the effect of physical nonlinearity it is recommended to register and compare the eigenfrequencies. There may be problems to identify the nonlinearity from the eigenmodes because very precise measurements of the shapes of the eigenmodes are required.

In the process of nondestructive identification of nonlinearities the eigenmode of a statically loaded structure is excited. Then the shape of the eigenmode is registered by using the method of time-averaged projection moiré. From the obtained experimental image it is usually possible to determine which eigenmode is excited. The analysis is also performed for the same eigenmode of the unloaded structure. In both cases the eigenfrequencies are measured as well. Then the change of the eigenfrequency is calculated from the measured eigenfrequencies for the same eigenmode of the unloaded and the statically loaded structures. The effect of nonlinearity is determined on the basis of the obtained change of eigenfrequency.

The obtained results are used in the process of design of the elements of packages.

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