410. Application of software SPRING and method of complete bifurcation groups for the bifurcation analysis of nonlinear dynamical system

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Abstract. The article is devoted to the parametrical analysis of periodic and chaotic oscillations in the nonlinear dynamical systems. Description of the systematic approach to the construction of bifurcation diagrams is offered to your attention. Considered approach allows us to build complete bifurcation diagrams. These diagrams allow to find and investigate regimes, analysis of which is inaccessible by traditional methods. Such approach is used in the method of complete bifurcation groups and realized in the software SPRING. Description of method of complete bifurcation groups and results of using of this method for the parametric analysis of the nonlinear dynamic systems is presented in this article.

Keywords: numerical simulation, bifurcation, complete bifurcation diagram, bifurcation group, bifurcation map.

Introduction to the method of complete bifurcation group

Method of complete bifurcation groups allows to get complete bifurcation diagrams. These diagrams are contained more information, what traditional diagrams, and allows to use the systematic approaches to their analysis and investigation.

Method of complete bifurcation group consists of the solve of the following problems:

1. Search of all periodic regimes at some fixed values of parameters. It is necessary to find all stable and unstable, harmonic and subharmonic periodic solutions. The subharmonic solutions should be found up to the some order. At the case of numeral solution the task can be solved only with some resolution.
2. Using a method of continuation on parameter and a hypothesis about a continuity of a branch of a bifurcation diagram, need to construct branches of diagram corresponding to each found regime. In most cases on the branches of diagram corresponding to unstable regimes, the greater or smaller stable fragments will be found. Collection of branches of the bifurcation diagram, received as a result of series of bifurcations, we shall call as bifurcation group.
3. On the basis of method of complete bifurcation groups the systematic approach to the investigation of the nonlinear dynamic systems is possible. Such approach is implied by knowledge of typical structures of bifurcation groups of the nonlinear dynamical systems of different nature. We think, that amount of different topologies of bifurcation groups is limited.

Software SPRING

The software SPRING is developed in Riga Technical University in a group under guidance of professor M.V.Zakrzhevsky. SPRING contains realization of methods and approaches which many years was developed and used in this group [2].

SPRING is the instrument of research of the stationary (periodic and non-periodic, stable and unstable) regimes of oscillations in the nonlinear dynamical systems on the basis of numeral simulation of their behavior. Investigated object is a mathematical model, described on the basis of the system of ordinary differential equations, for which the operator of discrete mapping is defined, or directly as a discrete mapping equation. SPRING allows to investigate any nonlinear dynamic systems, containing the arbitrary set of continuous and piecewise forces, instantaneous influences and impacts.

In the SPRING six basic tools of research of the dynamical systems are offered [5]: *time histories tools* allows to calculate projections of phase trajectory and points of discrete mapping; *outline mapping tools* allows to construct points of discrete mapping of phase trajectories family with initial points on the contour in a phase plane, *steady state tools* allows to search and analysis of characteristics of the stable and unstable periodic regimes; *bifurcation diagram tools* allows to construct complete bifurcation diagrams, containing the branches of stable and unstable solutions, by the method of motion on a parameter; *bifurcation map tools* allows to construct bifurcation maps by the method of motion on a bifurcation border; *cell-to-cell mapping tools* allows to construct basins of attraction of the stationary regimes [3,4].

One of basic possibilities of software SPRING is application of method of complete bifurcation groups for the construction of complete bifurcation diagrams.

Incomplete and complete bifurcation diagrams

The traditional method of construction of bifurcation diagrams is based on the analysis of transitional processes at the different values of the variable parameter. In this simplest realization the calculation of transient is executed in a current of some time and then a few last points of discrete mapping is saved on a graphic. If these points have same coordinates, it is a periodic regime with the period of excitation force P1. If points repeats oneself with some period, it is the subharmonic regime of proper order PN (for example regime P3). And if points do not repeats oneself even upon termination of transient, we can tell about chaotic or almost periodic attractor. Saving on one graph the results of such analysis, we will get the simplest bifurcation diagram (fig. 1).

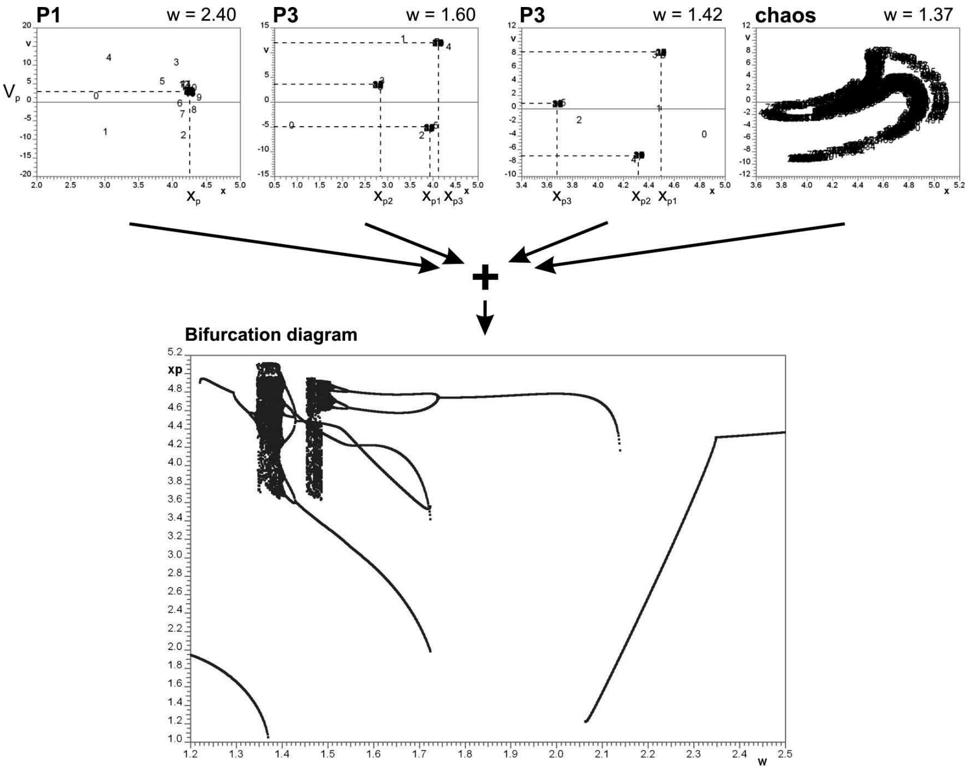

Fig. 1. Traditional bifurcation diagram constructed by the method of brute force

Obvious lacks of such approach are:

1. it is practically impossible to find the regimes with the small area of attraction or which is stable in the small area of parameter;
2. in the case of multistability it is difficult to show all branches of bifurcation diagram;
3. absence of system does not allow to do conclusions about structure of bifurcation diagram and to extend these conclusions to others dynamical systems.

The bifurcation diagram, which corresponds to mentioned above traditional diagram, contains two bifurcation groups for the basic regime P1 and subharmonic regime P3 (fig. 2). The unstable branches of diagram allow to unite the separate stable areas of diagram in a single structure (group). Construction of complete bifurcation groups with all bifurcation and unstable regimes allows to find all stable fragments of the diagram (even with the small areas of attraction and ranges of parameter in which regime is stabile). A complete bifurcation diagram, in a difference from traditional, has more systematical structure. Knowledge of typical structures of bifurcation groups facilitates a bifurcation analysis, as allows to extends conclusions about the structure of bifurcation diagrams of the studied dynamical system to the bifurcation diagrams of the new system.

Application of method of complete bifurcation groups consists of two stages: search of all periodic (stable and unstable) attractors at the fixed values of parameters and construction of bifurcation group on the basis of every founded periodic solution. The software SPRING gives the set of tools for the solving of these tasks.

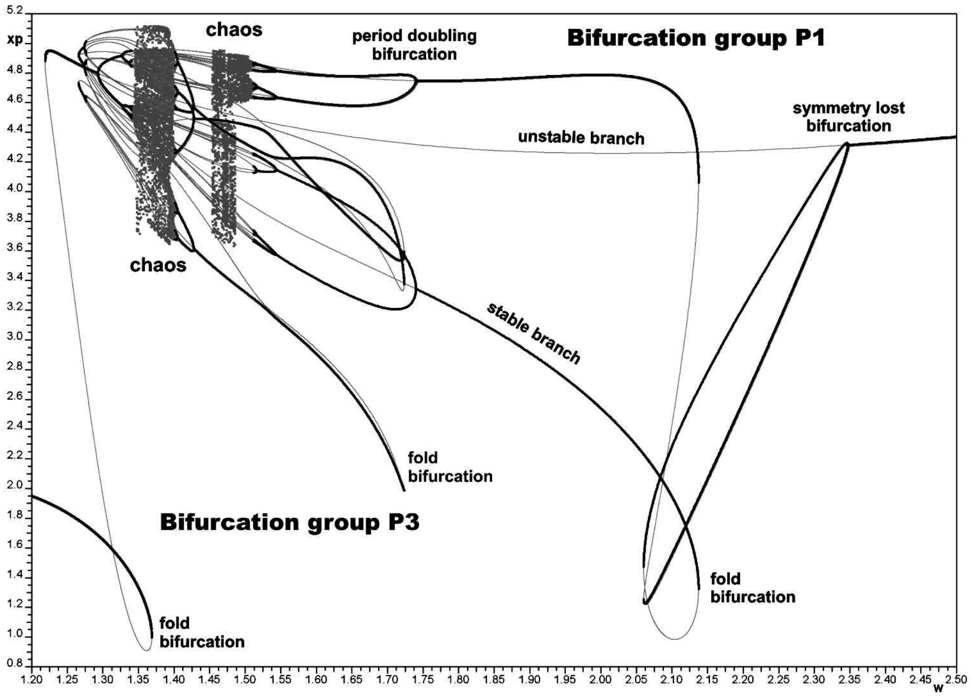

Fig. 2. Bifurcation diagram constructed by the method of complete bifurcation group

Construction of complete bifurcation diagram

Let's pass to illustration of construction of complete bifurcation diagram for one of a dynamical system. We will illustrate application of method of complete bifurcation groups by the construction of bifurcation diagram for the driven damped oscillator with cubic elastic force and harmonic excitation (eq. 1). The results of numeral simulation at the constant parameters b = 0.2 and h = 30 and at the change of excitation force frequency w = 1.2 .. 2.5 will be used. For an analysis the points of discrete periodic mapping with the period of excitation force (Poincare mapping) are used.

$$\ddot{x} + b\dot{x} + (x + x^3) = h\cos(\omega t + \varphi) \quad (1)$$

In the considered example the search of all periodic regimes with the period of excitation force P1 and subharmonic regimes of the second and third orders was carried out. 19 periodic regimes are found (fig. 3). From them only one stable. Five periodic regimes with the period of excitation force (A-E). From them three symmetric (A, B, E) and two mutually symmetric twins (C, D). Six subharmonic regimes P2 – three pair of twins (F-G, H-I, J-K). Eight subharmonic regimes P3 – two symmetric (L and M) and three pair of twins (N-O, P-Q, R-S).

Fixed points of the periodic regimes on a Poincare plane are in the basin of attraction of two stationary regimes – A and chaos. Saddle node unstable point is disposed on the areas border, other unstable points are inside the basin of attractions of chaotic attractor (fig. 4).

A further analysis allows us to divide all of solutions into three bifurcation groups: bifurcation group of the basic regime P1 (points A-E, J, K), bifurcation group of island type regime P2 (points F-I) and two bifurcation groups of island type regimes P3 (points L-O and points P-S). For a construction a bifurcation group is enough to know one fixed point of any regime of this group. Thus it is possible to start construction of four complete bifurcation groups.

To construct bifurcation group of base regime P1 (fig. 5), begin from the point A and, using the method of continuation on a parameter, move along branch of bifurcation diagram of the basic regime P1. We pass through fold bifurcation; get on an unstable branch; pass the point of the regime B; pass through fold bifurcation and restore stability; loss of symmetry bifurcation leads to occurrence of two stable twins; period doubling bifurcation leads to the loss of stability of these regimes and birth of the subharmonic regimes P2. The remaining regimes of bifurcation group are got: the unstable symmetric regime P1 (E), two unstable mutually symmetric regimes P1 (C, D) and two mutually symmetric unstable regimes P2 (J, K). The cascade of period doubling leads to appearance of infinite number of the unstable periodic regimes (UPI – unstable periodic infinitium). Such bifurcation structure results in appearance of the chaotic regimes. This regime can be unstable and exists only during transients or can be stable and exists as attractor. In considered case at w = 1.35 UPI leads to birth of chaotic attractor. Further development of bifurcation group occurs under the similar scenario: the cascade of period doubling bifurcations results in disappearance of UPI, the asymmetrical regimes P1 pass through two folds, then symmetry of regime is restored through symmetry loss bifurcation. As a result we receive bifurcation diagram of bifurcation group P1. The received structure of bifurcation group is typical and meets in many dynamical systems.

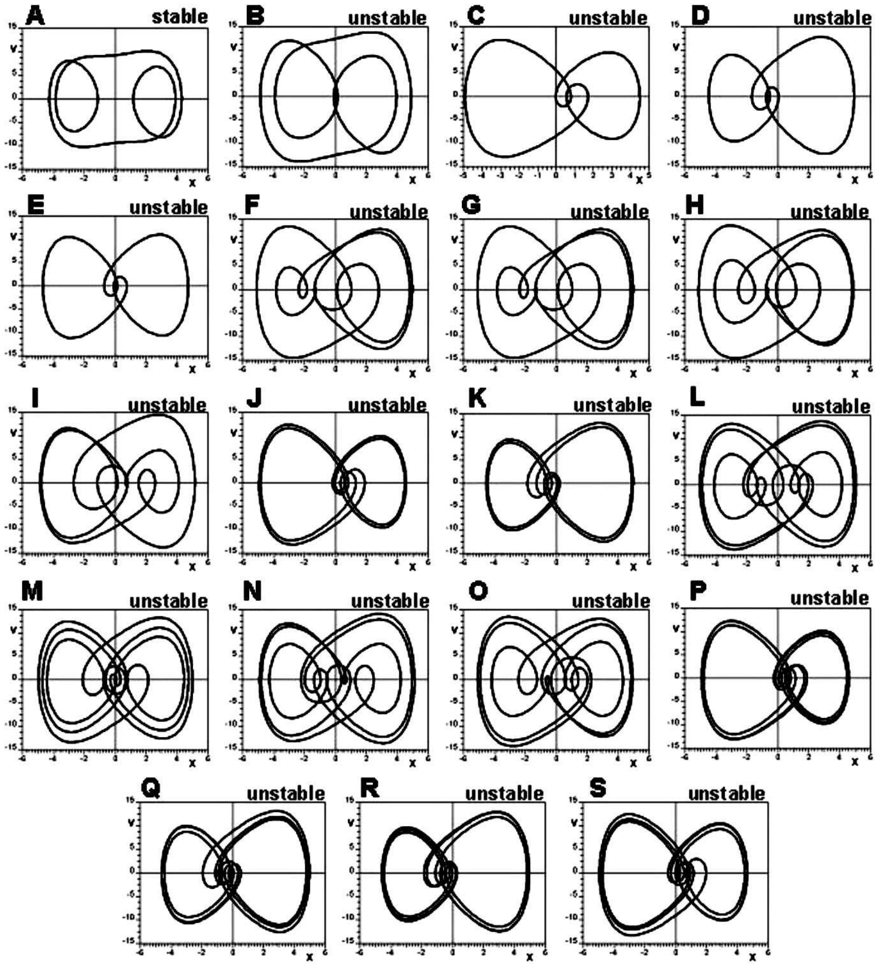

Fig. 3. Periodic solutions of investigated system at w = 1.35 and h = 30. Basic regimes with period of excitation force P1 (A-E), subharmonical regimes with period 2 (F-K) and subharmonical regimes with period 3 (L-S)

The second bifurcation group of island type regime P2 is other example of typical structure of bifurcation diagram (fig. 6). A group is mainly made by the unstable branches of UPI regimes. A group is both-side limited by fold bifurcations. After these bifurcations the regimes of this group becomes stable. But this stability is saved only in the very small range of parameter w. Then the periodic regimes of group lose stability and as a result of series of period doubling bifurcations UPI is born. We name the stable regimes of this bifurcation group as rare attractors because of their small area of existence on parameter. All branches of this group (including rare attractors) can be constructed beginning from one of steady points of any regime of this group (F, G, H or I).

The third bifurcation group of island type regime P3 (fig. 7) has a structure analogical considered before bifurcation group of the basic regime P1. The construction of bifurcation group can begin from any of the found fixed points of the unstable regimes of this group (L, M, N or O). In a difference from the bifurcation group of the basic regime this group both-side limited by fold bifurcations. Further group development occurs under the similar scenario: a loss of a symmetry – a cascade of period doubling bifurcations – UPI – a cascade of period doubling bifurcations – a restore of a symmetry. There is rare attractor RA on one end of the received island, finding which began possible only due to knowledge of typical structure of bifurcation group.

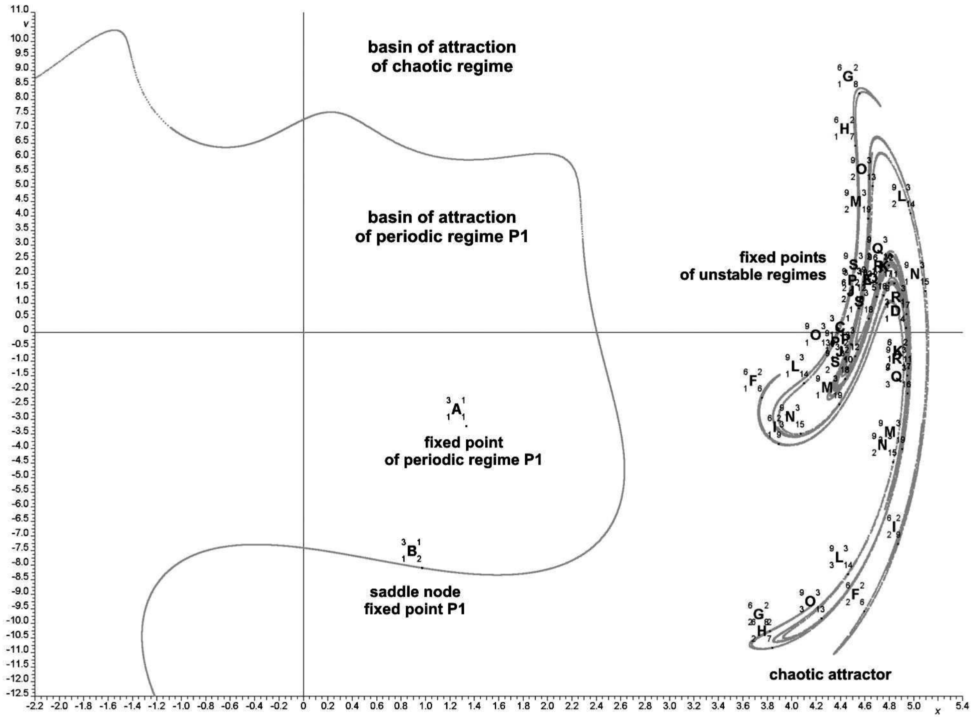

fig. 4. Fixed points of periodic solutions

The fourth bifurcation group of island type of regime P3 (fig. 8) has a structure analogical the bifurcation group of island type regime P2. The construction of group can begin from the points P, Q, R or S. As well as in the case of regime P2 the island is limited by fold bifurcations. There are rare attractors on the ends of island. Rare attractors lose stability as a result of serious of period doubling bifurcations. The same series of bifurcations lead to birth of UPI, which makes the basic part of bifurcation group.

Uniting all received bifurcation groups on one graph we will get a complete bifurcation diagram (fig. 9). On this bifurcation diagram, except mentioned before regimes P1 and P3, should be noted rare attractors and UPI.

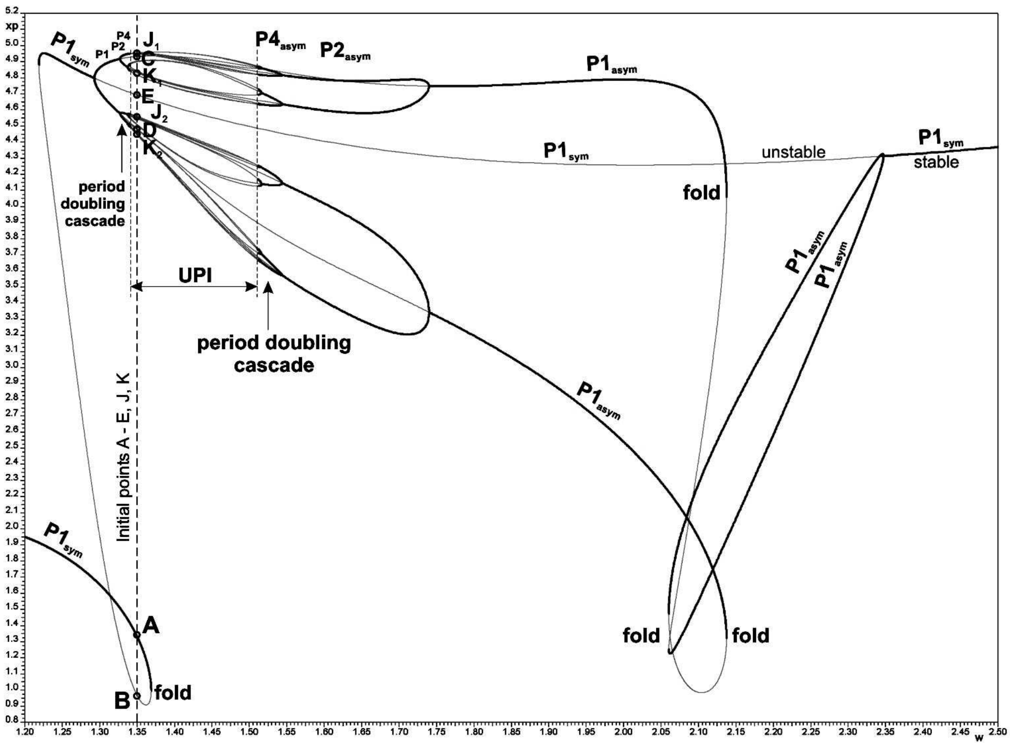

Fig. 5. Main bifurcation group of regime P1

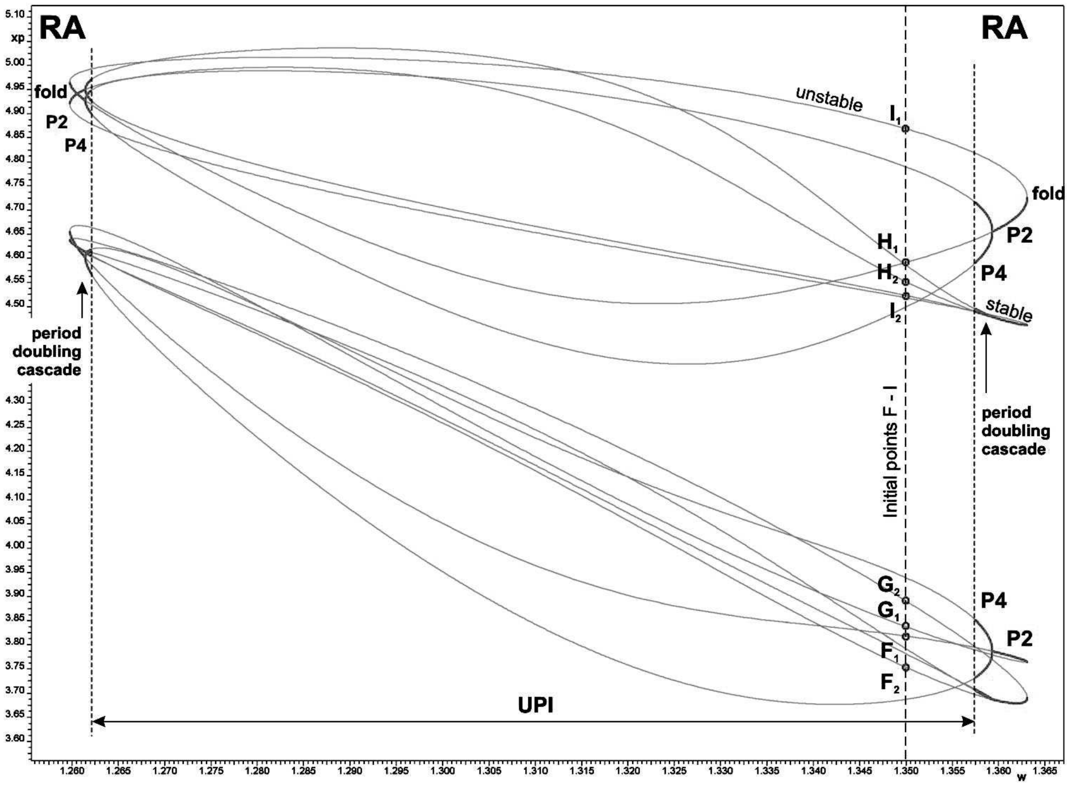

Fig. 6. Bifurcation group of island type regime P2

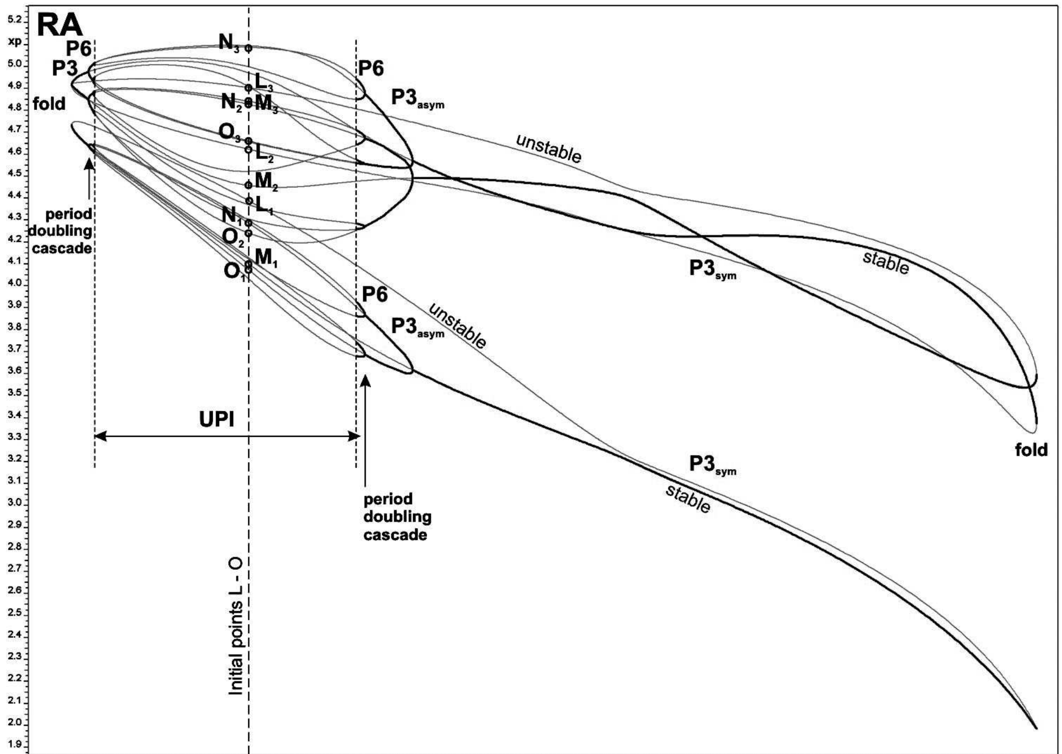

Fig. 7. Bifurcation group of first island type regime P3

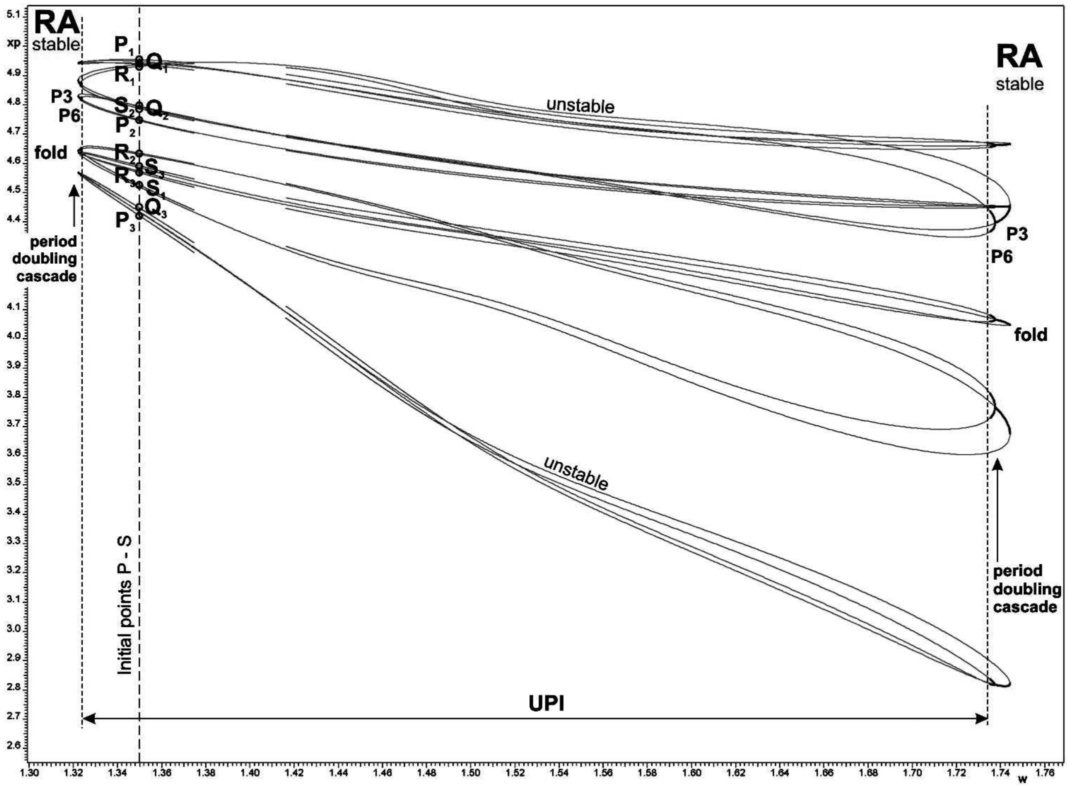

Fig. 8. Bifurcation group of second island type regime P3

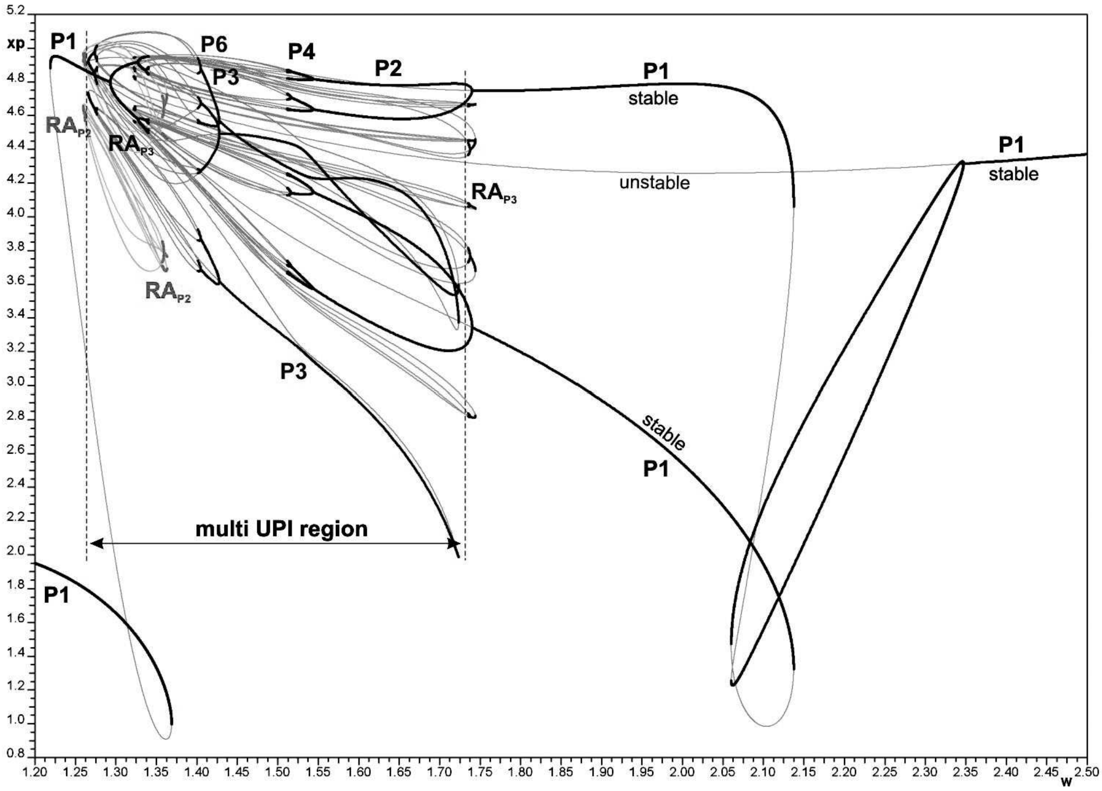

Fig. 9. Complete bifurcation diagram with four bifurcation groups

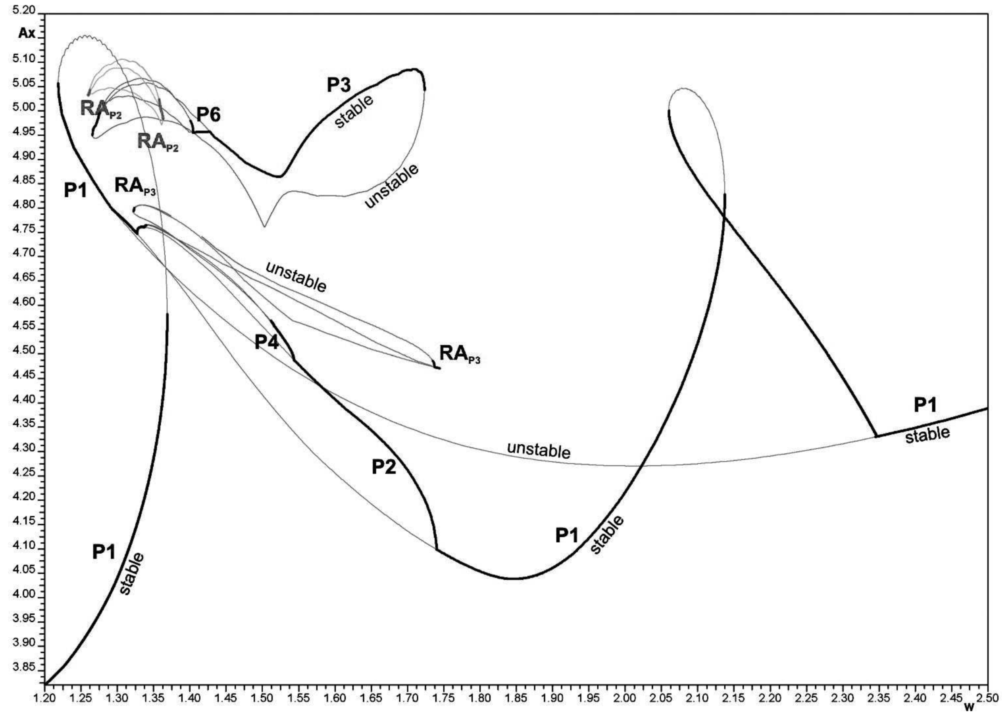

Fig. 10. Complete amplitude diagram with four bifurcation groups

The location of rare attractors is illustrated by a amplitude diagram (fig. 10). For example the regime RA_{P3} coexists with the basic regime P1, but has large amplitude. Such coexistence can prove as an unexpected (unpredictable) increase of amplitude of oscillations on some frequencies. Other rare attractors have considerable amplitudes also.

Bifurcation maps

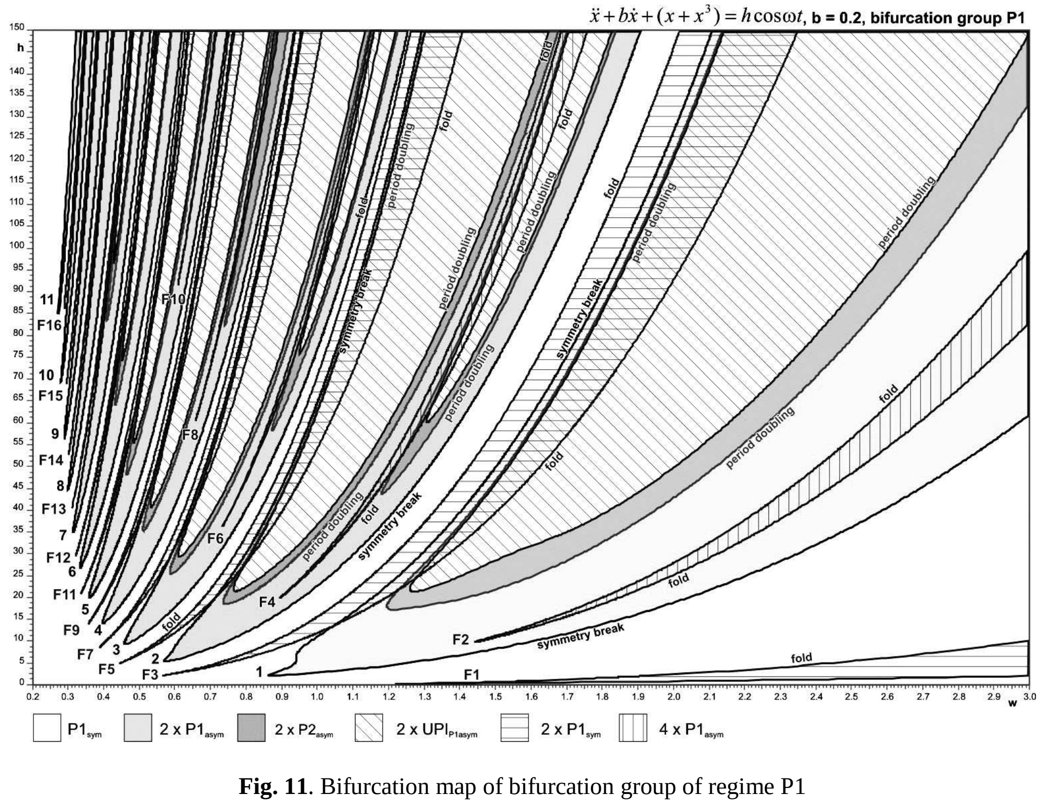

Fig. 11. Bifurcation map of bifurcation group of regime P1

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Other application of method of complete bifurcation groups is a method of bifurcation maps construction about which also it would be desirable to say a few words. Bifurcation map is built on the plane of two parameters and selects on this plane the areas of existence of the different stationary modes. The famous example of such map is a map, built by Yoshisuke Ueda for the system of Duffing [1]. Such map allows to determine the parameters of the system at which exists or does not exists interesting us regime of oscillations.

The construction of bifurcation maps can based on the method of continuation on parameter. Only in this case it must be extended on two parameters simultaneously (complex parameter). Taking for basis the bifurcation point of complete bifurcation group and varying two parameters of the system simultaneously it is possible to draw a line all of points of which correspond to bifurcations. Such algorithm is realized in the software Spring and named as BifJumper.

Taking for basis the bifurcation points of bifurcation group of the basic regime and applying the method of BifJumper, we will get a bifurcation map, shown on the figure 11. Are constructed bifurcation borders corresponding to bifurcation of a fold, loss of symmetry and period doubling. The sloping shading is select the areas of UPI with the chaotic behavior of the system. Choosing frequency and amplitude of excitation force from these areas it is possible to get chaos (possibly only transitional).

Conclusions

The presented new approach to the parametrical analysis of nonlinear dynamic systems on the basis of complete bifurcation groups gives new possibilities for construction of bifurcation diagrams and bifurcation maps [5]. This approach allows to receive complete bifurcation diagrams which, due to unstable branches, allow to find and show new regimes, which search and analysis by traditional methods is inconvenient. The method of complete bifurcation groups is represented as the perspective approach for investigating and systematization of bifurcation diagrams of various dynamical systems. Its advantages are especially obvious at research of such phenomena as rare attractors, UPI and chaos.

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