

403. Why can we detect the chaos?

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Abstract. A well known chaotic mapping in symbol space is a shift mapping. However, other chaotic mappings in symbol space exist too. The basic change is to consider the process (physical or social phenomenon) not only at a set of times which are equally spaced, say at unit time apart (a shift mapping), but at a set of times which are not equally spaced, say if we cannot fix unit time (an increasing mapping). Hence we regard x_t as being the flow of discrete signals when t is restricted to values $0, 1, 2, \dots$ but $x_{f(t)}$ the detection of these signals. Such interpretation simulates the observation. Our results reveal why we can detect chaos even our experiment is not shared in strict equally spaced time intervals. This as every mathematical treatment leads to a rigorous definition of chaos. We restrict ourselves with symbol space A^ω , that is, we consider one sided infinite sequences $x_0, x_1, \dots, x_t, \dots$ with elements from a fixed set ($\forall t x_t \in A$). Our results is proved for such space, namely, the increasing mapping $f_\omega : A^\omega \rightarrow A^\omega$ is chaotic in the set A^ω , where

$$f_\omega(x) = x_{f(0)}x_{f(1)}x_{f(2)} \dots x_{f(i)} \dots, \quad i \in N, \quad x \in A^\omega, \quad 0 < f(0) \text{ and } \forall i \forall j [i < j \Rightarrow f(i) < f(j)].$$

Keywords: alphabet, infinite and finite sequences (or words), prefix metric, infinite symbol space, chaotic map, increasing mapping, dense orbit.

1 Introduction

The Chaotic dynamics has been hailed as the third great scientific revolution of the 20th century, along with relativity and quantum mechanics. The explosion of interest in nonlinear dynamical systems has led to the development of new mathematics. Chaotic and random behavior of deterministic systems is now understood to be an inherent feature of many nonlinear systems.

The basic goal of the theory of dynamical systems is to understand the eventual or asymptotic behavior of an iterative process. If the process is a discrete process such as the iteration of a function, then the theory hopes to understand the eventual behavior of the points

$$x, f(x), f^2(x), \dots, f^n(x), \dots$$

as n becomes large. That is, dynamical systems ask to somewhat nonmathematical sounding question: where do points go and what do they do when they get there? In this article, we will attempt to answer this question at least

partially for one of the simplest classes of dynamical systems, functions of a single variable in symbolic space.

The technique of characterizing the orbit structure of a dynamical system via infinite sequences of "symbols" is known as *symbolic dynamics*. Symbolic dynamics were first introduced by Emil Artin in 1924, in the study of Artin billiards [14].

The first exposition of symbolic dynamics as an independent subject was given by Morse and Hedlund [11], 1938). They showed that in many circumstances such a finite description of the dynamics is possible. Other ideas in symbolic dynamics come from the data storage and transmission. D.Lind and B.Marcus in 1995 have published first general textbook [7] on symbolic dynamics and its applications to coding. This book and B.P.Kitchens ([6], 1998) give a good account of the history of symbolic dynamics and its applications.

A well known chaotic mapping in symbol space is a shift mapping ([5], [6], [7], [12], [13]). However, other chaotic mappings in symbol space exist too. The basic

change is to consider the process (physical or social phenomenon) not only at a set of times which are equally spaced, say at unit time apart (a shift mapping), but at a set of times which are not equally spaced, say if we cannot fixed unit time (an increasing mapping).

There is a philosophy of modeling in which we study idealized systems that have properties that can be closely approximated by physical systems. The experimentalist takes the view that only quantities that can be measured have meaning. This is a mathematical reality that underlies what the experimentalist can see.

Our results reveal why we can detect chaos even our experiment is not shared in strict equally spaced time intervals.

The article is structured as follows. It starts with preliminaries concerning notations and terminology that is used in the paper followed by a definition of the chaotic mapping. The increasing mapping is considered in Section 4, furthermore, it is proved that this map is chaotic. Some non-chaotic mappings in the infinite symbol space are investigated in Section 6 too.

Much of what many researchers consider dynamical systems has been deliberately left out of this text. For example, we do not treat continuous systems or differential equations at all. For this reason the Section 5 is devoted to some interpretations.

2 Preliminaries

The section presents the notation and terminology used in this paper. Terminology comes from combinatorics on words (for example, [9] or [10]).

We give some notations at first:

$$\overline{k, n} = \{k, k+1, \dots, n\}, k \leq n \text{ and } k, n \in \{0, 1, 2, \dots\},$$

Z - the set of integers, $Z_+ = \{x | x \in Z \ \& \ x > 0\}$,

$$\mathbb{N} = Z_+ \cup \{0\}.$$

From now on A will denote a finite *alphabet*, i.e., a finite nonempty set $\{a_0, a_1, a_2, \dots, a_n\}$ and the elements are called *letters*. We assume that A contains at least two symbols. By A^* we will denote the set of all finite sequences of letters, or finite *words*, this set contains empty word (or sequence) λ too. $A^+ = A^* \setminus \{\lambda\}$. A word $\omega \in A^+$ can be written uniquely as a sequence of letters as $\omega = \omega_1 \omega_2 \dots \omega_l$, with $\omega_i \in A$, $1 \leq i \leq l$. The integer l is called the *length* of ω and denoted $|\omega|$. The length of λ is 0.

An extension of the concept of finite word is obtained by considering infinite sequences of symbols over a finite set. *One-sided* (from left to right) infinite sequence or word, or simply *infinite word*, over A is any total map $\omega: \mathbb{N} \rightarrow A$. The set A^ω contains all infinite words. $A^\infty = A^* \cup A^\omega$.

If the word $u = u_0 u_1 u_2 \dots \in A^\infty$, where $u_0, u_1, u_2, \dots \in A$, then finite word $u_0 u_1 u_2 \dots u_n$ is called the *prefix* of u of length $n+1$. The empty word λ is assumed to be the prefix of u of length 0.

$$\text{Pref}(u) = \{\lambda, u_0, u_0 u_1, u_0 u_1 u_2, \dots, u_0 u_1 u_2 \dots u_n, \dots\}$$

(that is, $\text{Pref}(u)$ is the set of all prefixes of word u).

Secondly we introduce in A^∞ a metric d as follows.

Definition 2.1. Let $u, v \in A^\infty$. The mapping $d: A^\infty \times A^\infty \rightarrow R$ is called a *metric* or *prefix metric* in the set A^∞ if

$$d(u, v) = \begin{cases} 2^{-m}, & u \neq v, \\ 0, & u = v, \end{cases}$$

where $m = \max\{|\omega| \mid \omega \in \text{Pref}(u) \cap \text{Pref}(v)\}$

It is easy to prove that the function d is a metric (see, for example, [10]).

3 Definition of chaotic mapping

The term “chaos” in reference to functions was first used in Li and Yorke's paper “Period three implies chaos” ([8], 1975). We use the following definition of R.Devaney [2]. Let (X, ρ) be metric space.

Definition 3.1 ([2]). The function $f: X \rightarrow X$ is chaotic if

- the periodic points of f are dense in X ,
- f is topologically transitive,
- f exhibits sensitive dependence on initial conditions.

At first we note

Definition 3.2. The function $f: X \rightarrow X$ is *topologically transitive* on X if

$$\forall x, y \in X \ \forall \varepsilon > 0 \ \exists z \in X \ \exists n \in \mathbb{N}:$$

$$\rho(x, z) < \varepsilon \ \& \ \rho(f^n(z), y) < \varepsilon.$$

Definition 3.3. The function $f: X \rightarrow X$ *exhibits sensitive dependence on initial conditions* if

$$\exists \delta > 0 \ \forall x \in X \ \forall \varepsilon > 0 \ \exists n \in \mathbb{N}:$$

$$\rho(x, y) < \varepsilon \ \& \ \rho(f^n(x), f^n(y)) > \delta.$$

Definition 3.4. Let $A, B \subseteq X$ and $A \subseteq B$. Then A is *dense* in B if

$$\forall x \in B \ \forall \varepsilon > 0 \ \exists y \in A: \rho(x, y) < \varepsilon.$$

Devaney's definition is not the only classification of a chaotic map. For example, another definition can be found in [12]. Also mappings with only one property - sensitive dependence on initial conditions - frequently are considered as chaotic (see, [4]). Banks, Brooks, Cairns,

Davis and Stacey [1] have demonstrated that for continuous functions, the defining characteristics of chaos are topological transitivity and the density of periodic points.

Theorem 3.1. Let A be an infinite subset of metric space X and $f: A \rightarrow A$ to be continuous. If f is topologically transitive on A and the periodic points of f are dense in A , then f is chaotic on A .

It means that we can not check up exhibits sensitive dependence on initial conditions of mapping. This property follows from others.

See also [5] chapter 11.

Theorem 3.2. Let A be a subset of a metric space X and $f: A \rightarrow A$. If the periodic points of f are dense in A and there is a point whose orbit under iteration of f is dense in the set A , then f is topologically transitive on A .

Therefore we conclude

Corollary 3.1. Continuous function f is chaotic in infinite metric space X , if following conditions are met:

- 1) the periodic points of f are dense in the set X ,
- 2) either there exists a point orbit of which by map f is dense in the set X , either f is topologically transitive in the set X .

4 Increasing mapping

Let $f_\omega(x) = x_{f(0)}x_{f(1)}x_{f(2)}\dots x_{f(i)}\dots, i \in \mathbb{N}, x \in A^\omega$.

In this case the function f is called *the generator function* of mapping f_ω .

Definition 4.1. A function $f: \mathbb{N} \rightarrow \mathbb{N}$ is called *positively increasing function* if

$$0 < f(0) \text{ and } \forall i \forall j [i < j \Rightarrow f(i) < f(j)].$$

The mapping $f_\omega: A^\omega \rightarrow A^\omega$ is called *increasing mapping* if its generator function $f: \mathbb{N} \rightarrow \mathbb{N}$ is positively increasing.

Example 4.1. For example, let's take a look at positively increasing function: $\forall x \in \mathbb{N}: f(x) = 3x + 1$. It is clear that every positively increasing function is increasing function in ordinary sense but not conversely. The function $f(x) = 3x, x \in \mathbb{N}$, is increasing function in ordinary sense. Since $0 = f(0)$ the function f is not positively increasing function.

If we consider $f(x) = 3x + 1$ as generator function, then the corresponding generated mapping is increasing, it is $f_\omega: A^\omega \rightarrow A^\omega$, where

$$\forall s = s_0s_1s_2\dots \in A^\omega: f_\omega(s) = s_1s_4s_7\dots s_{3i+1}\dots, i \in \mathbb{N}.$$

The well known shift map is increasing mapping in one-sided infinite symbol space A^ω , in this case the generator function is a positively increasing function $f: \mathbb{N} \rightarrow \mathbb{N}$, where $f(x) = x + 1$.

Let K be a set. The iterations of mapping $g: K \rightarrow K$ we define inductively:

- (i) $g^0 = \mathbf{I}$ (identical mapping);
- (ii) $g^{n+1} = gg^n$.

Lemma 4.1. If $f: \mathbb{N} \rightarrow \mathbb{N}$ is a positively increasing function, then $\forall n \in \mathbb{Z}_+ f^n$ is a positively increasing function and $\forall i \forall j \exists n: f^n(i) > j$.

Proof. We make the proof by induction on number of iterations n . If $n = 1$, then f is positively increasing function by conditions of Definition 4.2. Now we assume that f^n is positively increasing function: $f^n: \mathbb{N} \rightarrow \mathbb{N}$ and fulfils the conditions $0 < f^n(0)$ and $\forall i \forall j [i < j \Rightarrow f^n(i) < f^n(j)]$. We must show that f^{n+1} is positively increasing function too.

Since $0 < f^n(0)$ (first inductual assumption) then by second assumption $f(0) < f(f^n(0)) = f^{n+1}(0)$. Since f is positively increasing function then $0 < f^{n+1}(0)$.

If $i < j$, then by inductual assumption $f^n(i) < f^n(j)$. Since f is positively increasing function then $f(f^n(i)) = f^{n+1}(i) < f(f^n(j)) = f^{n+1}(j)$, and therefore we have proved that $\forall n \in \mathbb{Z}_+ f^{n+1}$ is a positively increasing function.

Now we will prove the second part of Lemma. At first we notice that:

$$\forall i \in \mathbb{N}: i < f(i).$$

The case $i = 0$ follows from definition of positively increasing function. We assume inductively that inequality $i < f(i)$ is true for every fixed $i \in \mathbb{N}$ and prove that this inequality is true for $i + 1$ as well. Since $i < i + 1$, by second condition of positively increasing function a condition of inequality is fulfilled: $f(i) < f(i + 1)$. Since by inductual assumption $i < f(i)$ and $f(i) \in \mathbb{N}$ then $i + 1 \leq f(i)$. Summary it means that $i + 1 \leq f(i) < f(i + 1)$.

Since f is positively increasing function then

$$i < f(i) < f(f(i)) = f^2(i) < \dots \\ \dots < f^{n-1}(i) < f(f^{n-1}(i)) = f^n(i).$$

We know that $f(i) \in \mathbb{N}$, therefore $f^n(i) > i + n$. We conclude if $n \geq 1$ and $n > j - 1$, then $f^n(i) > j$.

Definition 4.2. The mapping $g : K \rightarrow K$ has a *dense orbit* in the set K if there exist a point $x \in K$ such that the set $\{g^k(x) | k \in \mathbb{N}\}$ dense in the set K .

Theorem 4.1. For increasing mapping $f_\omega : A^\omega \rightarrow A^\omega$ exists a dense orbit in the set A^ω .

Proof. Let $\beta : \mathbb{N} \rightarrow A^+$ is a freely chosen bijection.

(For example, if $A = \{0,1\}$ and

$$A^+ = \{0,1,00,01,10,11,000,001,010,\dots\}, \text{ then}$$

$$\beta(0) = 0, \beta(1) = 1, \beta(2) = 00, \beta(3) = 01, \beta(4) = 10,$$

$$\beta(5) = 11, \beta(6) = 000, \dots$$

We define inductively a sequence of words

$$u_0, u_1, \dots, u_n, \dots \in A^+ \tag{1}$$

such that

$$\forall i [|u_i| < |u_{i+1}| \wedge u_i \in \text{Pref}(u_{i+1})].$$

We also define a sequence of integers

$$k_0, k_1, \dots, k_m, \dots$$

The definition is as follows:

1) We choose $u_0 = \beta(0)$ and $k_0 = 0$.

2) Let $a \in A, \beta(1) = \beta_{10}\beta_{11} \dots \beta_{1l_1}$ and $u_0 = \beta(0) = u[0]u[1] \dots u[s_0]$, where all $\beta_{1i}, u[j]$ are letters of alphabet A . By Lemma 4.1 $\exists k_1 f^{k_1}(0) > s_0$.

The word $u_1 = u_1[0]u_1[1] \dots u_1[s_1]$, where $s_1 = f^{k_1}(l_1)$, is defined following

$$u_1[j] = \begin{cases} u[j], & \text{if } j \in \overline{0, s_0}; \\ \beta_{10}, & \text{if } j = f^{k_1}(0); \\ \beta_{11}, & \text{if } j = f^{k_1}(1); \\ \dots & \dots \\ \beta_{1l_1}, & \text{if } j = f^{k_1}(l_1) = s_1; \\ a, & \text{other cases.} \end{cases}$$

3) We assume that $\beta_{n0}\beta_{n1} \dots \beta_{nl_n}$ and $u_{n-1} = u[0]u[1] \dots u[s_{n-1}]$, where all $\beta_{ni}, u[j]$ are letters

of alphabet A . By Lemma 4.1 $\exists k_n f^{k_n}(0) > s_{n-1}$. We define the word $u_n = u_n[0]u_n[1] \dots u_n[s_n]$, where $s_n = f^{k_n}(l_n)$, following

$$u_n[j] = \begin{cases} u[j], & \text{if } j \in \overline{0, s_{n-1}}; \\ \beta_{n0}, & \text{if } j = f^{k_n}(0); \\ \beta_{n1}, & \text{if } j = f^{k_n}(1); \\ \dots & \dots \\ \beta_{nl_n}, & \text{if } j = f^{k_n}(l_n) = s_n; \\ a, & \text{other cases.} \end{cases}$$

4) Since $\forall i u_i \in \text{Pref}(u_{i+1})$ and $\lim_{i \rightarrow \infty} |u_i| = \infty$ then there exists an infinite word

$$u \in A^\omega \text{ such that } u = \lim_{i \rightarrow \infty} u_i.$$

The orbit of word u is dense in the set A^ω . Let $\varepsilon > 0$. Then there exists m such that $2^{-m} < \varepsilon$. We assume that $x \in A^\omega$ and v is a prefix of word x of length m . Then there exists n such that $\beta(n) = v$. By construction of the sequence (1)

$$\beta(n) \in \text{Pref}(f^{k_n}(u)),$$

therefore distance $d(x, f^{k_n}(u)) \leq 2^{-m} < \varepsilon$.

Remark. If the set A is countable, then A^+ is countable too, therefore the proof of Theorem 4.1 does not change if the set A is countable.

Theorem 4.2. The periodic point set of increasing mapping $f_\omega : A^\omega \rightarrow A^\omega$ is dense in the set A^ω .

Proof. The proof is similar as for the Theorem 4.1. Let $\varepsilon > 0$. Then there exists m such that $2^{-m} < \varepsilon$. We assume that $x \in A^\omega$ and v is a prefix of word x of length m . We define inductively sequence of words

$u_0, u_1, \dots, u_n, \dots$ such that

$$\forall i [|u_i| < |u_{i+1}| \wedge u_i \in \text{Pref}(u_{i+1})].$$

1) We choose $u_0 = v$.

2) Let $s = m - 1$. We assume that $a \in A$ and $u_0 = u[0]u[1] \dots u[s]$, where all $u[j]$ are letters of the alphabet A . By Lemma 4.1 $\exists k f^k(0) > s$. We define the word $u_1 = u_1[0]u_1[1] \dots u_1[s_1]$, where $s_1 = f^k(s)$, following

$$u_1[j] = \begin{cases} u[j], & \text{if } j \in \overline{0, s}; \\ u[0], & \text{if } j = f^k(0); \\ u[1], & \text{if } j = f^k(1); \\ \dots & \dots \\ u[s], & \text{if } j = f^k(s) = s_1; \\ a, & \text{other cases.} \end{cases}$$

Since $f^k(0) > s$ then $f^{2k}(0) > f^k(s) = s_1$.

3) We assume that $u_{n-1} = u[0]u[1] \dots u[s_{n-1}]$, where $f^{nk}(0) > f^{(n-1)k}(s) = s_{n-1}$ and all $u_{n-1}[j]$ are letters of alphabet A . The word $u_n = u_n[0]u_n[1] \dots u_n[s_n]$, where $s_n = f^k(s_{n-1})$ is defined following

$$u_n[j] = \begin{cases} u[j], & \text{if } j \in \overline{0, s_{n-1}}; \\ u[0], & \text{if } j = f^{nk}(0); \\ u[1], & \text{if } j = f^{nk}(1); \\ \dots & \dots \\ u[s_{n-1}], & \text{if } j = f^{nk}(s_{n-1}) = s_n; \\ a, & \text{other cases.} \end{cases}$$

Since $f^{nk}(0) > f^{(n-1)k}(s) = s_{n-1}$ then $f^{(n+1)k}(0) > f^{nk}(s) = f^k(s_{n-1}) = s_n$.

4) Since $\forall i u_i \in \text{Pref}(u_{i+1})$ and $\lim_{i \rightarrow \infty} |u_i| = \infty$ then there exists an infinite word $u \in A^\omega$ such that $u = \lim_{i \rightarrow \infty} u_i$.

According to construction of u , it follows that $d(u; x) \leq 2^{-m} < \varepsilon$ and $f^k(u) = u$. We have proved that u is a periodic point of mapping f_ω such that the distance between u and x is less than ε .

Theorem 4.3. *The increasing mapping $f_\omega : A^\omega \rightarrow A^\omega$ is continuous in the set A^ω .*

Proof. We fix word $u \in A^\omega$ and $\varepsilon > 0$. We need to prove that there is $\delta > 0$ such that whenever $d(u; v) < \delta$, then $d(f_\omega(u), f_\omega(v)) < \varepsilon$.

We choose m such that $2^{-m} < \varepsilon/2$ and assume that $0 < \delta < 2^{-f(m)+1}$, where $f : \mathbb{N} \rightarrow \mathbb{N}$ is corresponding positively increasing function of f_ω . If $d(u; v) < \delta$, then by definition of prefix metric follows that $u_i = v_i$ for all $i=0,1,\dots,f(m)$.

From definition of increasing mapping

$$f_\omega(u) = u_{f(0)}u_{f(1)} \dots u_{f(m)} \dots$$

$$f_\omega(v) = v_{f(0)}v_{f(1)} \dots v_{f(m)} \dots$$

and $u_{f(i)} = v_{f(i)}$, $i = 0,1,\dots,m$, therefore

$$d(f_\omega(u), f_\omega(v)) \leq 2^{-(m+1)} < 2^{-m} < \varepsilon.$$

Theorem 4.4. *The increasing mapping $f_\omega : A^\omega \rightarrow A^\omega$ is chaotic in the set A^ω .*

Proof. The space A^ω is infinite space. Since the increasing mapping $f_\omega : A^\omega \rightarrow A^\omega$ is continuous (Theorem 4.3), there exists a dense orbit in the set A^ω (Theorem 4.1) and its periodic point set is dense in the set A^ω (Theorem 4.2) then by Corollary 3.1 follows that increasing mapping $f_\omega : A^\omega \rightarrow A^\omega$ is chaotic in the set A^ω .

5 Interpretations

(i) Let
$$x(t_0), x(t_1), \dots, x(t_n), \dots \tag{2}$$

be the flow of discrete signals. Suppose that we have the experimentally observed subsequence

$$x(T_0), x(T_1), \dots, x(T_n), \dots \tag{3}$$

If $T_0 = t_1, T_1 = t_2, \dots, T_n = t_{n+1}, \dots$ we have the shift map. Notice if we have the infinite word $x = x_0x_1 \dots x_n \dots$ instead of sequence (2), then we have respectively the

infinite word $y = y_0y_1 \dots y_n \dots$ instead of (3). Here $\forall t y_t = x_{t-1}$. Hence, we obtain the shift map $f(t) = t + 1$, namely, $y = f_\omega(x) = x_{f(0)}x_{f(1)}x_{f(2)} \dots x_{f(n)} \dots$

We do not claim the function $f(t) = t + 1$ is chaotic on the real line \mathbb{R} but we had proved that this function as a generator creates the chaotic map f_ω in the symbol space A^ω . We had proved something more, namely, every positively increasing function g as a generator creates the chaotic map g_ω in the symbol space A^ω . In other words, if we had detected in our experiment only subsequence $x(t_1), x(t_3), \dots, x(t_{2n-1}), \dots$ even then we can reveal chaotic behavior.

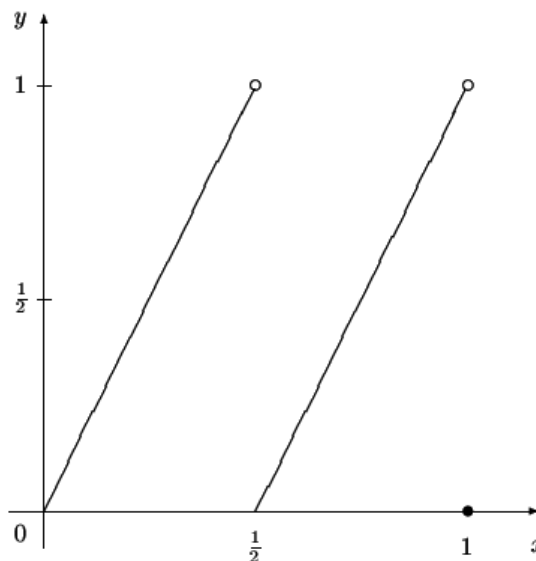


Fig. 1 $y = 2x \text{ mod } 1$.

(ii) Now we turn our attention to the interval $[0, 1]$ of the real line \mathbb{R} . Let us consider a dynamical system defined by the map

$$y = 2x \text{ mod } 1. \tag{4}$$

The key step is to recognize that because the slope of the graph (Fig. 1) is 2 everywhere, the action of (4) is trivial if the coordinates $x \in [0,1]$ are represented in the base 2. Let x have the binary expansion $x = 0.x_0x_1 \dots x_n \dots$

with $\forall t x_t \in \{0,1\}$. It is easy to see that the next iterate will be $y = (x_0.x_1x_2 \dots x_n \dots) \text{ mod } 1 = 0.x_1x_2 \dots x_n \dots$. Thus, the base 2 expansion of y is obtained by dropping the leading digit in the expansion of x . This map is chaotic in the interval $[0, 1]$ (see, e.g., [3]). Notice the map

$$h : \{0,1\}^\omega \rightarrow [0,1] : x_0x_1 \dots x_n \dots \mapsto 0.x_1x_2 \dots x_n \dots$$

is the topological semi-conjugacy (see, e.g., [12]) from the shift map to the action of (4). Similarly we can obtain the chaotic map in the interval $[0, 1]$ from every increasing mapping $f_\omega : A^\omega \rightarrow A^\omega$. For example, let the positively increasing function $\alpha : \mathbb{N} \rightarrow \mathbb{N}$

$$\alpha(i) = \begin{cases} i + 1, & i \in \{0,1\}, \\ i + 2, & i \geq 2. \end{cases}$$

Then $\alpha_\omega : \{0,1\}^\omega \rightarrow \{0,1\}^\omega$ is the increasing mapping, and therefore it is chaotic in the set $\{0,1\}^\omega$. Now we can obtain the chaotic function in the interval $[0, 1]$, namely, the function

$$E_3(x) = \begin{cases} 4x, & 0 \leq x < \frac{1}{16}, \\ 4x - \frac{1}{4}, & \frac{1}{16} \leq x < \frac{3}{16}, \\ 4x - \frac{1}{2}, & \frac{3}{16} \leq x < \frac{5}{16}, \\ 4x - \frac{3}{4}, & \frac{5}{16} \leq x < \frac{7}{16}, \\ 4x - 1, & \frac{7}{16} \leq x < \frac{8}{16}, \\ 4x - 2, & \frac{8}{16} \leq x < \frac{9}{16}, \\ 4x - \frac{9}{4}, & \frac{9}{16} \leq x < \frac{11}{16}, \\ 4x - \frac{5}{2}, & \frac{11}{16} \leq x < \frac{13}{16}, \\ 4x - \frac{11}{4}, & \frac{13}{16} \leq x < \frac{15}{16}, \\ 4x - 3, & \frac{15}{16} \leq x < 1, \\ 0, & x = 1. \end{cases}$$

The graph of the function $E_3(x)$ is sketched in Fig. 2.

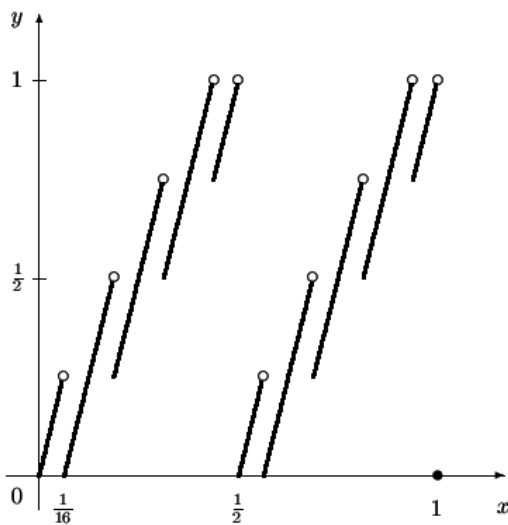


Fig. 2 $E_3(x)$.

6 Non-chaotic mappings

At first we prove one result about case that we had not considered in section 4.

Theorem 6.1. *If generator function f of mapping $f_\omega : A^\omega \rightarrow A^\omega$ is such that $f(0) = 0$, then the generated mapping f_ω is not topologically transitive in the set A^ω .*

Proof. We prove the opposite of topological transitivity:

$$\exists x \exists y \exists \varepsilon > 0 \forall z \forall n \in \mathbb{N} \quad (d(x, z) \geq \varepsilon \vee d(f_\omega^n(z), y) \geq \varepsilon).$$

We assume that $x = x_0x_1 \dots x_n \dots$ and $y = y_0y_1 \dots y_n \dots$ are chosen so that $x_0 \neq y_0$ and $\varepsilon = 1$. Let $z \in A^\omega$ be arbitrary. Two cases are possible:

1) $z_0 \neq x_0$, then by definition of prefix metric $d(z, x) = 2^0 = 1 = \varepsilon$;

2) if $z_0 = x_0$, then we can not state that $d(z, x) \geq \varepsilon$, but we have $\forall n \in \mathbb{N} \quad f_\omega^n(z) = z_{f^n(0)}z_{f^n(1)}z_{f^n(2)} \dots$

In this iteration $z_{f^n(0)} = z_0$, but $z_0 = x_0 \neq y_0$, therefore

$$d(f_\omega^n(z), y) = 2^0 = 1 = \varepsilon.$$

Corollary 6.1. *If generator function f of mapping $f_\omega : A^\omega \rightarrow A^\omega$ is such that $f(0) = 0$, then the generated mapping f_ω is not chaotic in the set A^ω .*

Similar as in Theorem 5.1 it is easy to show that generated mapping f_ω is not topologically transitive (also not chaotic) in the set A^ω if for corresponding generator function $\exists i \in \mathbb{N} \quad f(i) = i$.

At second we prove one broader result for very large class of functions that is not chaotic.

Theorem 6.2. *If generator function $f : \mathbb{N} \rightarrow \mathbb{N}$ of mapping $f_\omega : A^\omega \rightarrow A^\omega$ is not one-to-one function, then the mapping f_ω is not topologically transitive in the set A^ω .*

Proof. We assume that generator function $f : \mathbb{N} \rightarrow \mathbb{N}$ of mapping $f_\omega : A^\omega \rightarrow A^\omega$ is not one-to-one, then there exist two different numbers k and m ($k < m$) such that $f(k) = f(m)$.

We assume that $x = x_0x_1 \dots x_i \dots$ and $y = y_0y_1 \dots y_i \dots$ are chosen so that $y_k \neq y_m$ and $x_m \neq y_m$. We assume that $\varepsilon = 2^{-m}$. We choose $z \in A^\omega$. If $d(z, x) \geq \varepsilon = 2^{-m}$, then the proof is completed. If $d(z, x) < \varepsilon = 2^{-m}$, then from definition of prefix metric follows that $\forall i \in \overline{0, m} \quad z_i = x_i$. This means that $z_m = x_m \neq y_m$, therefore $d(z, y) \geq \varepsilon = 2^{-m}$. We have assumed that $f(k) = f(m)$ then

$$f_\omega(z) = z_{f^n(0)}z_{f^n(1)} \dots z_{f^n(k)} \dots z_{f^n(m)} \dots$$

and $z_{f(k)} = z_{f(m)}$. But $y_k \neq y_m$, therefore

$$1) \quad z_{f(k)} = z_{f(m)} \neq y_k \Rightarrow d(f_\omega(z), y) \geq 2^{-k} > 2^{-m} \text{ or}$$

$$2) \quad z_{f(k)} = z_{f(m)} \neq y_m \Rightarrow d(f_\omega(z), y) > 2^{-m}.$$

Similar for every n in iteration $f_\omega^n(z)$ the k -th and m -th symbols are equal and those are different from y_k or y_m , therefore $d(f_\omega(z), y) \geq 2^{-m}$.

Corollary 6.2. *If generator function $f : \mathbb{N} \rightarrow \mathbb{N}$ of mapping $f_\omega : A^\omega \rightarrow A^\omega$ is not one-to-one function, then the mapping f_ω is not chaotic.*

Corollary 6.3. *If $f_\omega : A^\omega \rightarrow A^\omega$ is chaotic map in the set A^ω , then generated function $f : \mathbb{N} \rightarrow \mathbb{N}$ of mapping f_ω is one-to-one.*

For example, the increasing function $f_\omega : A^\omega \rightarrow A^\omega$ is a special case for mapping with one-to-one generator function. But we remark that there exist non-chaotic mappings with one-to-one generator functions. For example, identical mapping

$$\forall x \in A^\omega \quad f_\omega(x) = x$$

is not chaotic mapping but its generator function $\forall i \in \mathbb{N} \quad f(i) = i$ is one-to-one.

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