402. Lyapunov quantities, limit cycles and strange behavior of trajectories in two-dimensional quadratic systems

Kuznetsov N. V.[a] **and Leonov G. A.**[b]
Faculty of Mathematics and Mechanics,
Saint-Petersburg State University, Universitetsky pr. 28,
Peterhoff, Saint-Petersburg, 198504, Russia
e-mail: [a]*kuznetsov@math.spbu.ru*, [b]*leonov@math.spbu.ru*

(Received:02 October; accepted: 02 December)

Abstract. The computation of Lyapunov quantities is closely connected with the important in engineering mechanics question of dynamical system behavior near to "safe" or "dangerous" boundary of the stability domain. In classical works for the analysis of system behavior near boundary of the stability domain was developed the method of Lyapunov quantities (or Poincare-Lyapunov constants), which determine system behavior in the neighborhood of the boundary. In the present work a new method for computation of Lyapunov quantities, developed for the Euclidian coordinates and in the time domain, is suggested and is applied to investigation of small limit cycles. The general formula for computation of the third Lyapunov quantity for Lienard system is obtained. Transformations between quadratic system and special type of Lienard system are described. The computation of large (normal amplitude) limit cycles for quadratic systems such that the first and second Lyapunov quantities are equal to zero and the third one is not equal zero were carried out. In these computations the quadratic system is reduced to the Lienard equation and by the latter the two-dimensional domain of parameters, corresponding the existence of four limit cycles (three "small" and one "large") was evaluated. This domain extends the domain of parameters obtained for the quadratic system with four limit cycles due to Shi in 1980.

Keywords: Lyapunov quantity, Poincare-Lyapunov constant, period constant, (normal amplitude) large limit cycle, small limit cycle, two-dimensional autonomous systems, Lienard equation, quadratic system

Introduction

The computation of Lyapunov quantities is closely connected with the important in engineering mechanics question of dynamical system behavior near to boundary of the stability domain. Followed by the work of Bautin [1], one differs "safe" or "dangerous" boundaries, a slight shift of which implies a small (invertible) or noninvertible changes of system status, respectively. Such changes correspond, for example, to scenario of "soft" or "hard" excitations of oscillations, considered by Andronov [2].

In classical works of Poincare [3] and Lyapunov [4] for the analysis of system behavior near boundary of the stability domain was developed the method of computation of so-called Lyapunov quantities (or Poincare-Lyapunov constants), which determine a system behavior in the neighborhood of boundary. This method also permits us effectively to study the bifurcation of birth of small cycles [1, 6–15], which correspond in mechanics to small vibrations.

In the present work the method of Lyapunov quantities is applied to investigation of small limit cycles. A new method for computation of Lyapunov quantities, developed for the Euclidian coordinates and in the time domain, is suggested. The general formula for computation of the third Lyapunov quantity for Lienard system is obtained.

Also, the computer modeling of large (normal amplitude) limit cycles are carried out. The transformation of quadratic system to a special type of Lienard system is used for investigation of large limit cycles. For this type of Lienard systems there is obtained a domain on the plane of two parameters of system, which the systems with three small and one large cycles correspond to (around two different stationary point). In our computer experiments the effects of trajectories "flattening", that make the computational modeling difficult, are observed.

Methods of calculation of Lyapunov quantities

For computation of Lyapunov quantities one usually consider a sufficiently smooth two-dimensional system with two purely imaginary eigenvalues of lineal part of

$$\frac{dx}{dt} = -y + f(x,y), \qquad \frac{dy}{dt} = x + g(x,y). \tag{1}$$

Here $x,y \in R$ and the functions $f(\cdot,\cdot)$ and $g(\cdot,\cdot)$ have continuous partial derivatives of (n)-st order in the open neighborhood U of radius R_U of the point (x,y)=(0,0). Suppose, the expansion of the functions f, g begins with the terms not lower than the second order and therefore we have

$$f(0,0) = g(0,0) = 0, \quad \frac{df}{dx}(0,0) = \frac{df}{dy}(0,0) = \frac{dg}{dx}(0,0) = \frac{dg}{dy}(0,0) = 0. \tag{2}$$

By assumption on smoothness in the neighborhood U we have

$$\begin{aligned} f(x,y) &= \sum_{k+j=2}^{n} f_{kj} x^k y^j + o((|x|+|y|)^n) = \\ &= f_n(x,y) + o((|x|+|y|)^n), \\ g(x,y) &= \sum_{k+j=2}^{n} g_{kj} x^k y^j + o((|x|+|y|)^n) = \\ &= g_n(x,y) + o((|x|+|y|)^n). \end{aligned} \tag{3}$$

The study of limit cycles and Lyapunov quantities of two-dimensional dynamical systems was stimulated by as purely mathematical problems (the center-and-focus problem, Hilbert's sixteenth problem, and isochronous centers problem) as many applied problems (the oscillations of electronic generators and electrical machines, the dynamics of populations) [1–21]. The problems of greater dimension (when there are two purely imaginary roots and the rest are negative) can be reduced to two-dimensional problems with the help of procedure, proposed by Lyapunov [4].

At present, there exist different methods for determining Lyapunov quantities and the computer realizations of these methods, which permit us to find Lyapunov quantities in the form of symbolic expressions, depending on expansion coefficient of the right-hand sides of equations of system (see., for example, [3-10, 15] and others). These methods differ in complexity of algorithms and compactness of obtained symbolic expressions. The first method for finding Lyapunov quantities was suggested by Poincare [3]. This method consists in sequential constructing time-independent holomorphic integral for approximations of the system. Further, different methods for computation, which use the reduction of system to normal forms, was developed (see, for example, [6, 10]).

Another approach to computation of Lyapunov quantities is related with finding approximations of solution of the system. So, a classical approach [4] it is used changes for reduction of turn time of all trajectories to a constant (as, for example, in the polar system of coordinates) and procedures for recurrent construction of solution approximations.

In the works [12,15] a new method of computation of Lyapunov quantities is suggested which based on constructing approximations of solution (as a finite sum in powers of degrees of initial data) in the original Euclidean system of coordinates and in the time domain. The advantages of given method are due to its ideological simplicity and visualization power. This approach can also be applied to the problem of distinguishing of isochronous center since it permits us to find out approximation of time of trajectory "turn" (time constants) depend upon initial data [7,9,21].

The first and second Lyapunov quantities have been computed in the 40-50s of last century [1,23]. The third Lyapunov quantity was computed in terms of f_{ij} and g_{ij} in [14,15] and its expression occupies more then four pages and the expression for the fourth Lyapunov quantity occupies 45 pages.

Note that for reduction of symbolic expression and simplification of analysis of system, special transformations of system to complex variables [6,10,21] are often used.

Calculation of Lyapunov quantities by approximation of system integral.

Following the classical work [3,4], we consider a problem of computation of Lyapunov quantities by constructing the time independent integral $V(x,y)$ for system (**1**).

Since $V_2(x,y) = \frac{(x^2+y^2)}{2}$ is an integral of system of the first approximation and for the right-hand side of system smoothness condition (2) is satisfied, then in certain small neighborhood of zero state we seek the approximation of integral in the form

$$V(x,y) = \frac{x^2+y^2}{2} + V_3(x,y) + \ldots + V_{n+1}(x,y) \tag{4}$$

Here $V_k(x,y)$ are the following homogeneous polynomials

$$V_k(x,y) = \sum_{i+j=k} V_{i,j} x_i y_j \qquad k = 3,\ldots,n+1$$

with the unknown coefficients $V_{i,j}$. By (3) for the derivative of $V(x,y)$ in virtue of system (1) we have

$$\dot V(x,y) = \frac{\partial V(x,y)}{\partial x}\left(-y+\sum_{k+j=2}^{n} f_{kj}x^k y^j\right)+ \frac{\partial V(x,y)}{\partial y}\left(x+\sum_{k+j=2}^{n} g_{kj}x^k y^j\right)+ o((|x|+|y|)^{n+1}). \quad (5)$$

The coefficients of the forms V_k can always be chosen in such a way that

$$\dot V(x,y) = w_1(x^2+y^2)^2 + w_2(x^2+y^2)^3+\ldots + o((|x|+|y|)^{n+1}). \quad (6)$$

Here w_i are expressions depending only on coefficients of the functions *f* and *g*.

Then sequentially determining the coefficients of the forms V_k for *k*=3,... (for that at each step it is necessary to solve a system of (*k*+1) linear equations), from (**5**) and (**6**) we obtain the coefficient w_m that is the first not equal to zero

$$\dot V(x,y) = w_m(x^2+y^2)^{m+1} + o((|x|+|y|)^{2m+2}).$$

The expression w_m is usually [7] called a Poincare-Lyapunov constant ($2\pi w_m$ — *m*th Lyapunov quantity). Let the additional conditions [8]

$$V_{2m,2m+2}+V_{2m+2,2m}=0, \qquad V_{2m,2m}=0$$

be satisfied. Then at the *k*th step of iteration the coefficients $\{V_{i,j}\}_{i+j=k}$ can be determined uniquely from the linear equations system via the coefficients $\{f_{ij}\}_{i+j<k}$, $\{g_{ij}\}_{i+j<k}$ and the coefficients $\{V_{i,j}\}_{i+j<k}$, determined at the previous steps of iteration.

Calculation of Lyapunov quantities for the Euclidian coordinates in the time domain.

Here the new method, developed for the Euclidian coordinates and in time domain, not requiring the reduction to normal form, is described. The advantages of this method are due to its ideological simplicity and a visualization power.

The first steps in the development of this method were made in the works [11–15] and some related with it results can be found in the work [22].

Here we assume that

$$f(\cdot,\cdot), g(\cdot,\cdot):\mathbb{R}\times\mathbb{R}\to\mathbb{R} \in \mathbb{C}^{(n+1)}(U). \quad (7)$$

The existence condition of (*n*+1)th partial derivatives with respect to *x* and *y* for *f* and *g* is used for simplicity of exposition and can be weakened.

Approximation of solutions. Further we will use a smoothness of the functions *f* and *g* and will follow the first Lyapunov method on finite time interval (see f.e. classical works [23, 24] and others).

Let *x*(*t*,*x*(0),*y*(0)), *y*(*t*,*x*(0),*y*(0)) be a solution of system (**1**) with the initial data

$$x(0)=0,\ y(0)=h \quad (8)$$

Denote

$$x(t,h)=x(t,0,h),\ y(t,h)=y(t,0,h).$$

Below a time derivative will be denoted by x' and $\dot x$.

Lemma 1 *A positive number $H\in(0,R_U)$ exists such that for all $h\in[0,H]$ the solution (x(t,h),y(t,h)) is defined for $t\in[0,4\pi]$.*

The validity of lemma follows from condition (**2**) and the existence of two purely imaginary eigenvalues of the matrix of linear approximation of system (**1**).

This implies [26] the following

Lemma 2 *If smoothness condition (**7**) is satisfied, then*

$$x(\cdot,\cdot),\ y(\cdot,\cdot)\in\mathbb{C}^{(n+1)}([0,4\pi]\times[0,H]) \quad (9)$$

Further we will consider the sufficient small initial data $h\in[0,H]$, a finite time interval $t\in[0,4\pi]$ and use a uniform boundedness of the solution (*x*(*t*,*h*),*y*(*t*,*h*)) and its mixed partial derivatives with respect to *h* and *t* up to the order (*n*+1) inc in the set $[0,4\pi]\times[0,H]$.

We apply now a well-known linearization procedure [27].

From Lemma 2 it follows that for each fixed *t* the solution of system can be represented by the Taylor formula

$$\begin{aligned} x(t,h) &= h\frac{\partial x(t,\eta)}{\partial\eta}\Big|_{\eta=0} + \frac{h^2}{2}\frac{\partial^2 x(t,\eta)}{\partial\eta^2}\Big|_{\eta=h\theta_x(t,h)} \\ &\quad 0\le\theta_x(t,h)\le 1, \\ y(t,h) &= h\frac{\partial y(t,\eta)}{\partial\eta}\Big|_{\eta=0} + \frac{h^2}{2}\frac{\partial^2 y(t,\eta)}{\partial\eta^2}\Big|_{\eta=h\theta_y(t,h)} \\ &\quad 0\le\theta_y(t,h)\le 1, \end{aligned} \quad (10)$$

Note that by Lemma 2 and relation (10), the functions

$$\frac{h^2}{2}\frac{\partial^2 x(t,\eta)}{\partial\eta^2}\Big|_{\eta=h\theta_x(t,h)},\quad \frac{h^2}{2}\frac{\partial^2 y(t,\eta)}{\partial\eta^2}\Big|_{\eta=h\theta_y(t,h)}$$

and their time derivatives are smooth functions of *t* and have the order of smallness *o*(*h*) uniformly with respect to *t* on a considered finite time interval $[0,4\pi]$.

Introduce the following denotations

$$\tilde x_{h^k}(t) = \frac{\partial^k x(t,\eta)}{\partial^k\eta}\Big|_{\eta=0},\quad \tilde y_{h^k}(t) = \frac{\partial^k y(t,\eta)}{\partial^k\eta}\Big|_{\eta=0}.$$

We shall say that the sums

$$x_{h^m}(t,h) = \sum_{k=1}^{m}\tilde x_{h^k}(t)\frac{h^k}{k!} = \sum_{k=1}^{m}\frac{\partial^k x(t,\eta)}{\partial\eta^k}\Big|_{\eta=0}\frac{h^k}{k!},$$

$$y_{h^m}(t,h) = \sum_{k=1}^{m}\tilde y_{h^k}(t)\frac{h^k}{k!} = \sum_{k=1}^{m}\frac{\partial^k y(t,\eta)}{\partial\eta^k}\Big|_{\eta=0}\frac{h^k}{k!}$$

are the *m*th approximation of solution of system with respect to *h*. Substitute representation (10) in system (**1**). Then, equating the coefficients of h^1 and taking into account (**2**), we obtain

$$\frac{d\tilde{x}_{h^1}(t)}{dt} = -\tilde{y}_{h^1}(t),\qquad \frac{d\tilde{y}_{h^1}(t)}{dt} = \tilde{x}_{h^1}(t). \tag{11}$$

Hence, by conditions on initial data (**8**) for the first approximation with respect to *h* of the solution ($x(t,h),y(t,h)$), we have

$$x_{h^1}(t,h) = \tilde{x}_{h^1}(t)h = -h\sin(t),\qquad y_{h^1}(t,h) = \tilde{y}_{h^1}(t)h = h\cos(t). \tag{12}$$

Similarly, to obtain the second approximation $(x_{h^2}(t,h), y_{h^2}(t,h))$, we substitute representation

$$x(t,h) = x_{h^2}(t,h) + \frac{h^3}{3!}\frac{\partial^3 x(t,\eta)}{\partial\eta^3}\Big|_{\eta=h\theta_x(t,h)},\qquad y(t,h) = y_{h^2}(t,h) + \frac{h^3}{3!}\frac{\partial^3 y(t,\eta)}{\partial\eta^3}\Big|_{\eta=h\theta_y(t,h)}. \tag{13}$$

in formula (3) for $f(x,y)$ and $g(x,y)$. Note that in expressions for *f* and *g* (denote their by $u^f_{h^2}$ and $u^g_{h^2}$, respectively) in virtue of (**2**) the coefficients of h^2 depend only on $\tilde{x}_{h^1}(t)$ and $\tilde{y}_{h^1}(t)$, i.e., by (**12**) they are known functions of time and are independent of the unknown functions $\tilde{x}_{h^2}(t)$ and $\tilde{y}_{h^2}(t)$. Thus, we have

$$f(x_{h^2}(t,h)+o(h^2), y_{h^2}(t,h)+o(h^2)) = u^f_{h^2}(t)h^2 + o(h^2),$$
$$g(x_{h^2}(t,h)+o(h^2), y_{h^2}(t,h)+o(h^2)) = u^g_{h^2}(t)h^2 + o(h^2).$$

Substituting (**13**) in system (**1**), for the determination of $\tilde{x}_{h^2}(t)$ and $\tilde{y}_{h^2}(t)$ we obtain

$$\frac{d\tilde{x}_{h^2}(t)}{dt} = -\tilde{y}_{h^2}(t) + u^f_{h^2}(t),\qquad \frac{d\tilde{y}_{h^2}(t)}{dt} = \tilde{x}_{h^2}(t) + u^g_{h^2}(t). \tag{14}$$

Lemma 3 *For solutions of the system*

$$\frac{d\tilde{x}_{h^k}(t)}{dt} = -\tilde{y}_{h^k}(t) + u^f_{h^k}(t),\qquad \frac{d\tilde{y}_{h^k}(t)}{dt} = \tilde{x}_{h^k}(t) + u^g_{h^k}(t). \tag{15}$$

with the initial data

$$\tilde{x}_{h^k}(0) = 0, \tilde{y}_{h^k}(0) = 0 \tag{16}$$

we have

$$\begin{aligned}
\tilde{x}_{h^k}(t) &= u^g_{h^k}(0)\cos(t) + \\
&+\cos(t)\int_0^t \cos(\tau)((u^g_{h^k}(\tau))' + u^f_{h^k}(\tau))d\tau + \\
&+\sin(t)\int_0^t \sin(\tau)((u^g_{h^k}(\tau))' + u^f_{h^k}(\tau))d\tau - u^g_{h^k}(t),\\
\tilde{y}_{h^k}(t) &= u^g_{h^k}(0)\sin(t) + \\
&+\sin(t)\int_0^t \cos(\tau)((u^g_{h^k}(\tau))' + +u^f_{h^k}(\tau))d\tau - \\
&-\cos(\tau)\int_0^t \sin(\tau)((u^g_{h^k}(\tau))' + u^f_{h^k}(\tau))d\tau.
\end{aligned} \tag{17}$$

The relations (**17**) can be verified by direct differentiation.

Repeating this procedure for the determination of the coefficients $\tilde{x}_{h^k}$ and $\tilde{y}_{h^k}$ of the functions $u^f_{h^k}(t)$ and $u^g_{h^k}(t)$, by formula (**17**) we obtain sequentially the approximations $(x_{h^k}(t,h), y_{h^k}(t,h))$ for k=1,...,n. For $h\in[0,H]$ and $t\in[0,4\pi]$ we have

$$\begin{aligned}
x(t,h) &= \\
&= x_{h^n}(t,h) + \frac{h^{n+1}}{(n+1)!}\frac{\partial^{n+1}x(t,\eta)}{\partial\eta^{n+1}}\Big|_{\eta=h\theta_x(t,h)} = \\
&= x_{h^n}(t,h) + o(h^n) = \sum_{k=1}^n \tilde{x}_{h^k}(t)\frac{h^k}{k!} + o(h^n),\\
y(t,h) &= \\
&= y_{h^n}(t,h) + \frac{h^{n+1}}{(n+1)!}\frac{\partial^{n+1}y(t,\eta)}{\partial\eta^{n+1}}\Big|_{\eta=h\theta_y(t,h)} = \\
&= y_{h^n}(t,h) + o(h^n) = \sum_{k=1}^n \tilde{y}_{h^k}(t)\frac{h^k}{k!} + o(h^n),\\
&0\le\theta_x(t,h)\le 1,\quad 0\le\theta_y(t,h)\le 1.
\end{aligned} \tag{18}$$

Here by Lemma 2

$$\tilde{x}_{h^k}(\cdot), \tilde{y}_{h^k}(\cdot) \in \mathbb{C}^n([0,4\pi]),\quad k=1,...,n \tag{19}$$

and the estimate $o(h^n)$ is uniform $\forall\, t\in[0,4\pi]$. From (**16**) and by the choice of initial data in (**11**) we obtain $x_{h^k}(0,h) = x(0,h) = 0,\ y_{h^k}(0,h) = y(0,h) = h,\quad k=1,...,n.$

Computation of Lyapunov quantities in the time domain. Consider for the initial datum $h\in(0,H]$ the time $T(h)$ of first crossing of the solution ($x(t,h),y(t,h)$) of the half-line $\{x=0, y>0\}$. Complete a definition (by continuity) of the function $T(h)$ in zero: $T(0)=2\pi$. Since by (**12**) the first approximation of solution crosses the half-line $\{x=0,y>0\}$ at the time 2π, then the crossing time can be represented as

$$T(h)=2\pi+\Delta T(h),$$

where $\Delta T(h)=O(h)$. We shall say that $\Delta T(h)$ is a residual of crossing time.

By definition of $T(h)$ we have

$$x(T(h),h)=0. \tag{20}$$

Since by (**9**), $x(\cdot,\cdot)$ has continuous partial derivatives with respect to either arguments up to the order n inclusive and $\dot{x}(t,h)=\cos(t)h+o(h)$), by the theorem on implicit function, the function $T(\cdot)$ is n times differentiable. It is possible to show that $T(h)$ is also differentiable n times in zero. By the Taylor formula we have

$$T(h)=2\pi+\sum_{k=1}^{n}\tilde{T}_k h^k+o(h^n), \tag{21}$$

where $\tilde{T}_k=\frac{1}{k!}\frac{d^k T(h)}{dh^k}$ (called period constant [7,9,20]). We shall say that the sum

$$\Delta T_k(h)=\sum_{j=1}^{k}\tilde{T}_j h^j \tag{22}$$

is the kth approximation of the residual of the time $T(h)$ of the crossing of the solution $(x(t,h),y(t,h))$ of the half-line $\{x=0,\ y>0\}$. Substituting relation (**21**) for $t=T(h)$ in the right-hand side of the first equation of (**18**) and denoting the coefficient of h^k by $\tilde{x}_k$, we obtain the series $x(T(h),h)$ in terms of powers of h:

$$x(T(h),h)=\sum_{k=1}^{n}\tilde{x}_k h^k+o(h^n). \tag{23}$$

In order to express the coefficients $\tilde{x}_k$ by the coefficients $\tilde{T}_k$ of the expansion of residual of crossing time we assume that in (**18**) $t=2\pi+\tau$:

$$x(2\pi+\tau,h)=\sum_{k=1}^{n}\tilde{x}_{h^k}(2\pi+\tau)\frac{h^k}{k!}+o(h^n). \tag{24}$$

By smoothness condition (**19**) we have

$$\tilde{x}_{h^k}(2\pi+\tau)=\tilde{x}_{h^k}(2\pi)+\sum_{m=1}^{n}\tilde{x}^{(m)}_{h^k}(2\pi)\frac{\tau^m}{m!}+o(\tau^n),$$
$$k=1,\ldots,n.$$

Substitute this representation in (**24**) for the solution $x(2\pi+\tau,h)$ for $\tau=\Delta T(h)$, and bring together the coefficients of the same exponents h. Since $(\Delta T(h))^n=O(h^n)$, by (**20**) and taking into account (**21**) for $T(h)$, we obtain

$$h:0=\tilde{x}_1=\tilde{x}_{h^1}(2\pi),$$
$$h^2:0=\tilde{x}_2=\tilde{x}_{h^2}(2\pi)+\tilde{x}'_{h^1}(2\pi)\tilde{T}_1,$$
$$h^3:0=\tilde{x}_3=\tilde{x}_{h^3}(2\pi)+\frac{1}{2}\tilde{x}'_{h^2}(2\pi)\tilde{T}_1+\tilde{x}'_{h^1}(2\pi)\tilde{T}_2+$$
$$+\frac{1}{2}\tilde{x}''_{h^1}(2\pi)\tilde{T}_1^2,$$
$$\ldots\ldots\ldots$$
$$h^n:0=\tilde{x}_n=\ldots$$

From the above we sequentially find $\tilde{T}_j$. The coefficients $T_{k=1,\ldots,n-1}$ can be determined sequentially since the expression for $\tilde{x}_k$ involves only the coefficients $T_{m<k}$ and the factor $\tilde{x}'_{h^1}(2\pi)$ multiplying T_{k-1} is equal to -1.

We apply a similar procedure to determine the coefficients $\tilde{y}_k$ of the expansion

$$y(T(h),h)=\sum_{k=1}^{n}\tilde{y}_k h^k+o(h^n).$$

Substitute the representation

$$\tilde{y}_{h^k}(2\pi+\Delta T(h))=\tilde{y}_{h^k}(2\pi)+\sum_{k=1}^{n}\tilde{y}^{(m)}_{h^k}(2\pi)\frac{\Delta T(h)^m}{m!}+$$
$$+o(h^n),\quad k=1,\ldots,n$$

in the expression

$$y(2\pi+\Delta T(h),h)=\sum_{k=1}^{n}\tilde{y}_{h^k}(2\pi+\Delta T(h))\frac{h^k}{k!}+o(h^n).$$

Equating the coefficients of the same exponents h, we obtain the following relations

$$h:\ \tilde{y}_1=\tilde{y}_{h^1}(2\pi),$$
$$h^2:\tilde{y}_2=\tilde{y}_{h^2}(2\pi)+\tilde{y}'_{h^1}(2\pi)\tilde{T}_1,$$
$$h^3:\tilde{y}_3=\tilde{y}_{h^3}(2\pi)+\frac{1}{2}\tilde{y}'_{h^2}(2\pi)\tilde{T}_1+\tilde{y}'_{h^1}(2\pi)\tilde{T}_2+$$
$$+\frac{1}{2}\tilde{y}''_{h^1}(2\pi)\tilde{T}_1^2,$$
$$\ldots\ldots\ldots$$
$$h^n:\tilde{y}_n=\ldots$$

for the sequential determination of $\tilde{y}_{i=1,...,n}$. Here $\tilde{y}_{h_{k=1,...,n}}(\cdot)$ and $\tilde{T}_{k=1,..,n-1}$ are the obtained above quantities.

Thus, for n=2m+1 we sequentially obtained the approximations of the solution ($x(t,h)$,$y(t,h)$) at the time $t=T(h)$ of the first crossing of the half-line $\{x=0,y>0\}$ accurate to $o(h^{2m+1})$ and the approximation of the time $T(h)$ itself accurate to $o(h^{2m})$. If in this case $\tilde{y}_k = 0$ for k=2,..,2m, then $\tilde{y}_{2m+1}$ is called the mth Lyapunov quantity L_m. Note, that, according to the Lyapunov theorem, the first nonzero coefficient of the expansion $\tilde{y}_i$ is always of an odd number and for sufficiently small initial data h the sign of $\tilde{y}_i$ (of the Lyapunov quantity) designates a qualitative behavior (winding or unwinding) of the trajectory ($x(t,h)$,$y(t,h)$) on plane [4].

Lyapunov quantities for Lienard equation.

We apply now stated above two method for Lienard equation. Assuming in (**1**)

$$f(x,y) \equiv 0, \frac{dg(x,y)}{dy} = g_{x1}(x), g(x,0) = g_{x0}(x), \frac{dg_{x0}}{dx}(0) = 0,$$

we obtain the following system

$$\begin{aligned} \dot{x} &= -y, \\ \dot{y} &= x + g_{x1}(x)y + g_{x0}(x), \end{aligned} \tag{25}$$

Let be $g_{x1}(x) = g_{11}x + ..., g_{x0}(x) = g_{11}x^2 + ...$ Then

$$L_1 = -\frac{\pi}{4}(g_{20}g_{11} - g_{21}).$$

If $g_{21} = g_{20}g_{11}$, then $L_1 = 0$ and

$$L_2 = \frac{\pi}{24}(3g_{41} - 5g_{20}g_{31} - 3g_{40}g_{11} + 5g_{20}g_{30}g_{11}).$$

If $g_{41} = \frac{5}{3}g_{20}g_{31} + g_{40}g_{11} - \frac{5}{3}g_{20}g_{30}g_{11}$, then $L_2 = 0$

and

$$\begin{aligned} L_3 = -\frac{\pi}{576}(&70g_{20}^3g_{30}g_{11} + 105g_{20}g_{51} + 105g_{30}^2g_{11}g_{20} + \\ &+ 63g_{40}g_{31} - 63g_{11}g_{40}g_{30} - 105g_{30}g_{31}g_{20} - \\ &- 70g_{20}^3g_{31} - 45g_{61} - 105g_{30}g_{11}g_{20} + 45g_{60}g_{11}). \end{aligned}$$

The computation of Lyapunov quantities by means of two different analytic methods with applying the modern software tools of symbolic computing permits us to show that the formulas, obtained for Lyapunov quantities, are correct.

Transformation between quadratic system and the Lienard system.

Consider the Lienard equation

$$\ddot{x} + F(x)\dot{x} + G(x) = 0 \tag{26}$$

and the equivalent system

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -F(x)y - G(x), \end{aligned} \tag{27}$$

where

$$\begin{aligned} F(x) &= (Ax+B)x|x+1|^{q-2}, \\ G(x) &= (C_1x^3 + C_2x^2 + C_3x + 1)x\frac{|x+1|^{2q}}{(x+1)^3}. \end{aligned} \tag{28}$$

We have the following results [27;14].

Lemma 4 *Suppose, for the coefficients* A, B, C_1, C_2, C_3, q *of equation (**26**) the relations*

$$\begin{aligned} &\frac{(B-A)}{(2q-1)^2}\left((1-q)B + (3q-2)A\right) = \\ &= 2C_2 - 3C_1 - C_3, \end{aligned} \tag{29}$$

$$\begin{aligned} &\frac{(B-A)}{(2q-1)^2}\left(B + 2(q-1)A\right) = \\ &= C_2 - 2C_1 - 1, \end{aligned} \tag{30}$$

*are satisfied. Then by a nonsingular change, equation (**26**) can be reduced to the quadratic system*

$$\begin{aligned} \dot{x} &= p(x,y) = a_1x^2 + b_1xy + \alpha_1x + \beta_1y, \\ \dot{y} &= q(x,y) = a_2x^2 + b_2xy + c_2y^2 + \alpha_2x + \beta_2y \end{aligned} \tag{31}$$

with the coefficients

$$b_1 = 1, \alpha_1 = 1, \beta_1 = 1, c_2 = -q, \alpha_2 = -2, \beta_2 = -1,$$

$$\begin{aligned} a_1 &= 1 + \frac{B-A}{2q-1}, \\ a_2 &= -(q+1)a_1^2 - Aa_1 - C_1, \\ b_2 &= -A - a_1(2q+1). \end{aligned} \tag{32}$$

Then by the above relations for the Lyapunov quantities L_1 and L_2, we obtain the following in the point (0,0)

Lemma 5
if $L_1 = L_2 = 0$, $5A - 2Bq - 4B = 0$ *and* $A \neq B,\ AB \neq 0,\ q \neq \frac{1}{2}$ *then*

$$C_1 = (q+3)\frac{B^2}{25} - \frac{(1+3q)}{5},$$

$$C_2 = (15(1-2q)+3B^2)\frac{1}{25},$$

$$C_3 = \frac{3(3-q)}{5},$$

$$L_3 = -\frac{\pi B(q+2)(3q+1)[5(q+1)(2q-1)^2 + B^2(q-3)]}{20000}.$$

Thus, if the conditions of Lemma 5 and $L_3 \neq 0$, then by small disturbances of system we can obtain three "small" cycles around a zero equilibrium of system and seek "large" cycles on a plane of the rest two coefficients (B,q).

Lemma 6 *For $b_1 \neq 0$ system (**31**) can be reduced to the Lienard equation (**27**) with the functions*

$$F(x) = R(x)\,|\,\beta_1 + b_1 x\,|^q,\ \ G(x) = P(x)\,|\,\beta_1 + b_1 x\,|^{2q}\ .$$

Here $q = -\dfrac{c_2}{b_1}$

$$R(x) = -\frac{(b_1 b_2 - 2a_1 c_2 + a_1 b_1)x^2 + (b_2\beta_1 + b_1\beta_2 - 2\alpha_1 c_2 +}{(\beta_1 + b_1 x)^2}$$
$$\frac{+2a_1\beta_1)x + \alpha_1\beta_1 + \beta_1\beta_2}{(\beta_1 + b_1 x)^2},$$

$$P(x) = -(\frac{a_2 x^2 + \alpha_2 x}{\beta_1 + b_1 x} - \frac{(b_2 x + \beta_2)(a_1 x^2 + \alpha_1 x)}{(\beta_1 + b_1 x)^2} +$$
$$+\frac{c_2 (a_1 x^2 + \alpha_1 x)^2}{(\beta_1 + b_1 x)^3}).$$

Computer experiments.

The above results were applied to quadratic systems and the experiments for computing "large" cycles were performed. In these experiments the reduction of quadratic system to the Lienard equation of special form (**26**)-(**28**) was used and with its help a set of parameters B,q (Fig. 1), which correspond to the existence of "large" cycle, was estimated.

In Fig. 1 is shown a domain bounded by straight lines, which correspond to the lines of reversal sign of the third Lyapunov quantity. The curve C in the graph is a curve of the parameters B and q of the Lienard system, which correspond to parameters of quadratic system, such that for these parameters the results on the existence of four cycles were obtained in [29]. Since two Lyapunov quantities are equal to zero, by small disturbances it is possible to construct systems with four cycles for the considered domain of parameters: three small cycles round one equilibrium and one large cycle round another equilibrium.

These results were applied to quadratic systems and the experiments for computing "large" cycles were performed. Our experience of computations shows that it is practically impossible to trace "small" cycles in the neighborhood of equilibrium, where the zero and first Lyapunov quantities are equal to zero. However in a number of computer experiments we can distinctly see "large" cycles.

For example in Fig. 2 is shown a "large" cycle for the system

$$\dot{x} = 0.99x^2 + xy + x + y,$$
$$\dot{y} = -0.58x^2 - 0.17xy + 0.6y^2 - 2x - y,$$

the parameters of which correspond to the point P.

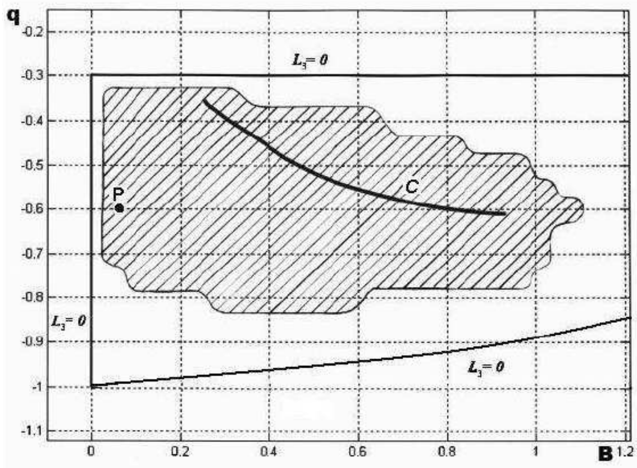

Fig.1. Domain of existence of "large" limit cycles

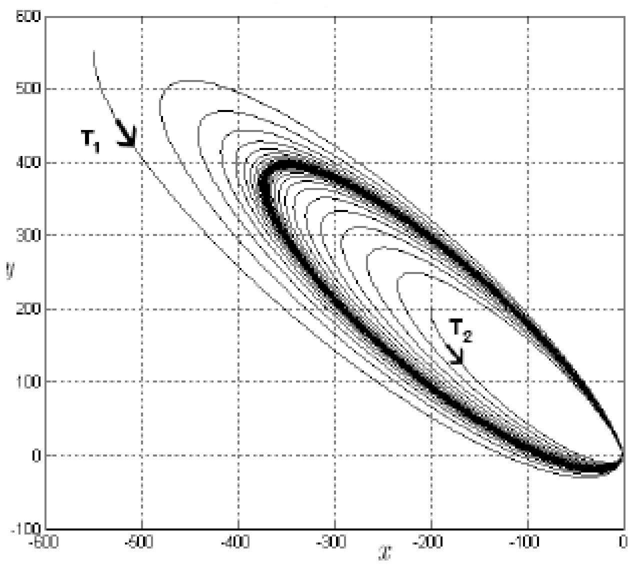

Fig.2. Stable limit cycle in quadratic system

In other experiments there occurs the interesting phenomenon of the existence of certain "domains of flattening", i.e. the "limit" points and "limit segments" of nonperiodic trajectories. The latter substantially involves the qualitative analysis of quadratic systems.

In Fig. 3 is shown the behavior of trajectory, which is "unwinded" from equilibrium and then extends at infinity, for the Lienard system with the functions

$$F(x) = \frac{(72x+60)x}{x+1},$$

$$G(x) = (\frac{2876}{5}x^3 + \frac{2175}{5}x^2 + \frac{6}{5}x + 1)\frac{x}{x+1}.$$

For this system the following condition $L_1 = L_2 = 0$ and $L_3 \neq 0$ is satisfied.

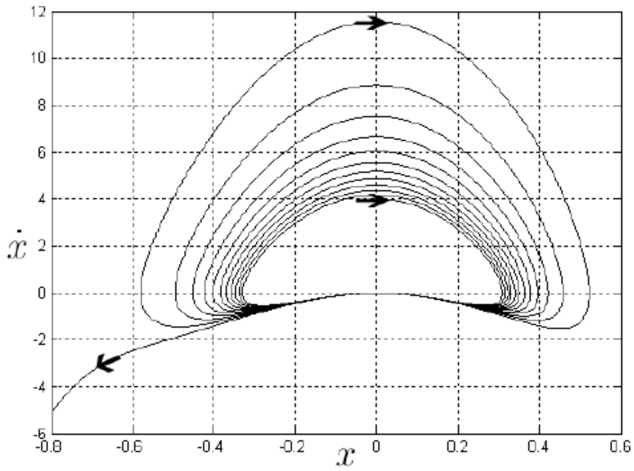

Fig. 3. Domains of flattening

Here we have the intense flattening of trajectories in a lower domain of graph. The trajectories in the domain of flattening very closely approaches to stationary point (the distance between the stationary point (0,0) and the domain of flattening is h=0.026).

Acknowledgment.

This work was partly supported by RFFI project 07-01-00151, Grants board of President RF MK-162.2007.1, NS-2387.2008.1 and Government of Saint-Petersburg

References

[1] Bautin N.N. Behavior of Dynamic Systems Near the Boundaries of the Stability Region, Gostekhizdat, Moscow-Leningrad, 1949 [in Russian]

[2] Andronov A. A., Vitt A. A., Khaikin S. E. Theory of oscillators, Oxford, New York: Pergamon Press, 1966

[3] Poincare H. Memoire sur les courbes definies par lesequations diffeentielles, J. de Mathemantiques Pures et Appliquees, (4) Vol. 1, pp. 167–244, 1885

[4] Lyapunov A. M. Stability of Motion. New York and London: Academic Press, 1966 (1892)

[5] Lloyd N. G. Limit cycles of polynomial systems – some recent developments. In New Direction in Dynamical Systems. Cambridge University Press, pp. 182–234, 1988

[6] Li J. Hilbert's 16th problem and bifurcation of planar polynomial vector fields, Int. J. Bifurcation Chaos 13, pp. 47—106, 2003

[7] Chavarriga J., Grau M. Some open problems related to 16th Hilbert problem, Sci. Ser. A Math. Sci. N.S., 9, pp. 1–26, 2003

[8] Lynch S. Symbolic computation of Lyapunov quantities and the second part of Hilbert's sixteenth problem. Chapter in the book Differential Equations with Symbolic Computations, pp. 1–26, 2005

[9] Gine J. On some problems in planar differential systems and Hilbert's 16th problem, Chaos, Solutions and Fractals, Vol. 31, pp. 1118–1134, 2007

[10] Yu P., Chen G. Computation of focus values with applications, Nonlinear Dynamics, 51(3), 2008

[11] Leonov G. A. Criterion for cycles existence in quadratic systems. Vestnik of St.Petersburg University , 3, pp. 31-41, 2007

[12] Kuznetsov N. V., Leonov G. A. Computation of the first Lyapunov quantity for the second-order dynamical system. Third IFAC Workshop Periodic Control System, 2007.

[13] Leonov G. A. Hilbert's 16th problem for quadratic system. New method based on a transformation to the Lienard equation. International journal of bifurcation and chaos, 18(3), pp. 877–884, 2008

[14] Kuznetsov N.V., Leonov G.A. Computation of Lyapunov quantities. 6th EUROMECH Nonlinear Dynamics Conference (*http://lib.physcon.ru/?item=1802*), 2008

[15] Leonov G. A., Kuznetsov N. V., Kudryashova E. V. Limit cycles of two dimensional systems. Calculations, proofs, experiments. Vestnik of St.Petersburg University, N3, pp.25–62, 2008

[16] Dumortier F., Llibre J., Artes J. Qualitative Theory of Planar Differential Systems, Springer, 2006

[17] Shilnikov L., Turaev D., Chua L. Methods of Qualitative Theory in Nonlinear Dynamics: Part 2, World Scientific, 2001

[18] Christopher C., Li C. Limit Cycles of Differential Equations, Birkhauser, 2007

[19] Perko L. Differential Equations and Dynamical Systems, Springer, 2001

[20] Roussarie R. Bifurcation of Planar Vector Fields and Hilbert's Sixteenth Problem, Birkhauser, 1998

[21] Gasull A., Guillamon A. and Manosa V. An explicit expression of the first Liapunov and period constants with applications. J. Math. Anal. Appl., 211, pp. 190–212, 1997

[22] Leonov G. A. Family of transversal curves for two-dimensional dynamical systems. Vestnik St.Petersburg University, 1(4), pp. 48–78, 2006

[23] Serebryakova N. Behavior of dynamical system with one degree of freedom near the points of boundaries of stability domain, where a safe boundary goes to the dangerous one [in Russian]. Izvestiya AN SSSR. Division of technical sciences, mechanics and engineering, 2, pp. 178–182, 1959

[24] Lefschetz S. Differential Equations: Geometric Theory. Interscience Publishers. New York. 1957

[25] Cesari L. Asymptotic Behavior and Stability Problems in Ordinary Differential Equations. Springer. Berlin. 1959

[26] Hartman P. Ordinary differential equation. John Willey & Sons. New York. 1964

[27] Leonov G.A., Kuznetsov N.V. Time-Varying Linearization and Perron effects, International journal of bifurcation and chaos. Vol. 17, Iss. 4, pp.1–29, 2007

[28] Leonov G.A. Two-Dimensional Quadratic Systems as a Lienard Equation, Differential Equations and Dynamical Systems, Vol. 5. N3/4, 1997

[29] Shi S.L. A concrete example of the existence of four limit cycles for plane quadratic systems, Sci. Sinica 23. pp. 153–158, 1980