401. Lame – manifolds in problems of synthesis of nonlinear oscillatory modes

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(Received:12 October; accepted: 02 December)

Abstract. In the article the theorem of the synthesis of multilinked systems with Lame - manifold is received. Offered synthesis allows to receive systems with the set geometry of stable limit cycles. The basic results of article are illustrated by the numerical modeling of processes of formation of limit cycles.

Keywords: invariance, asymptotic stability, stabilization, self-oscillations.

Introduction

The modern theory of synthesis of movement on the closed trajectories, systems of stabilization of movement in a direction of complication of geometry both trajectories and qualitative methods of synthesis actively develops. For example, many biological oscillatory processes carry relaxation character. Self-oscillatory modes arising there geometrically are defined by piecewise smooth limit cycles. In a radio engineering control of geometry of limit cycles is the basic problem at the decision of an extensive class of problems. The closed trajectories of movement of executive components are characterized by presence of sites close to the rectilinear in a robotics (Fig.1).

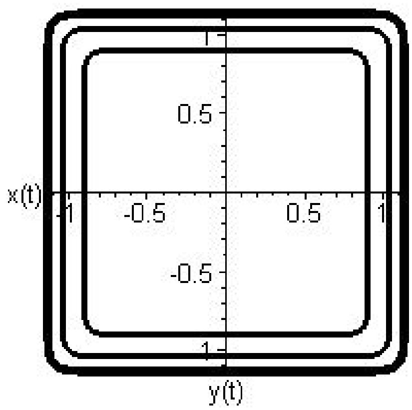

Fig. 1.

Movement trajectories are well approximated by closed curves of Lame. Closed curve of Lame is described by a following equation: $\frac{x_1^{2m}}{a_1^{2m}}+\frac{x_2^{2m}}{a_2^{2m}}=1$. At $m=1$ conservative oscillator, describing movement on such trajectories is linear. At $m>1$ it is received nonlinear conservative oscillator - oscillator of Lame which equations in phase variables have an obvious appearance $\dot{x}_1=-\frac{2m}{a_2^{2m}}x_2^{2m-1}$, $\dot{x}_2=\frac{2m}{a_1^{2m}}x_1^{2m-1}$.

Its trajectories at $m>1$ contain rectilinear sites of movement.

Problems of synthesis and stabilization of multilinked systems with set conservative parts of Lame are considered in work. Let's remind definition of invariant (integral) asymptotically stable manifold of system of the differential equations.

Definition 1*. Manifold Ω is called as integral manifold of system of the differential equations if system movements at the initial conditions defined on this manifold, contain in it at $t\in(-\infty,+\infty)$.*

Definition 2. **(Zubov V.I. [3])** *Integral manifold Ω is called as asymptotically stable, if:*

1. $\forall\varepsilon>0\exists\delta(\varepsilon)>0$, such, that at $\rho(\mathbf{X}_0,\Omega)<\delta$ inequality $\rho(\mathbf{X}(t,\mathbf{X}_0),\Omega)<\varepsilon$ is carried out at $t\geq t_0$;

2. $\forall\varepsilon>0\exists\delta(\varepsilon)>0$, such, that

$$\rho(\mathbf{X}(t,\mathbf{X}_0),\Omega)\underset{t\to+\infty}{\to}0,$$

where $\mathbf{X}(t,\mathbf{X}_0)$ − system movement, ρ − distance to integral manifold.

For such manifold the trajectories beginning on this manifold remain on it at $t\to+\infty$. If manifold of smooth system is closed and compact trajectories of system are unboundedly continued on it [3]. In our case such manifold is surface of Lame.

Besides, from existence and uniqueness of the decision of problem Cauchy, follows, that the trajectories of system beginning out of integral manifold cannot cross it at $t\to+\infty$. Hence, if this manifold is border of some domain the trajectories of system beginning inside of this domain, will remain in it at $t\in(-\infty,+\infty)$ [3].

Statement of a problem of synthesis of integral surfaces of Lame

Let's consider enclosed in R^{2n} smooth simply connected manifold $D^{2n} \subset R^{2n}$ with border $\partial \mathbf{D}_1^{2n} \cong \mathbb{S}_1^{2n-1}$, diffeomorfied to hyper sphere $\mathbb{S}_1^{2n-1}$, where

$$\partial D^{2n} \equiv \left\{ X \in R^{2n} \middle| 1 - \frac{x_1^2}{a_1^2} - \frac{x_2^2}{a_2^2} - \sum_{i=3}^{2n} \frac{x_i^{2m}}{a_i^{2m}} = 0 \right\}$$

$m \in Z^+$. Such borders are surfaces of Lame.

Let's consider the following system of the differential equations

$$\begin{cases} \dot{x}_1 = \pm\alpha_2 x_2, \\ \dot{x}_2 = \mp\alpha_1 x_1 + \mathrm{U}_2(\mathbf{X}), \\ \dot{x}_{2i-1} = \pm\alpha_{2i} x_{2i}^{2m-1}, \\ \dot{x}_{2i} = \mp\alpha_{2i-1} x_{2i-1}^{2m-1} + \mathrm{U}_{2i}(\mathbf{X}); \end{cases} \quad (1) \quad i = 2,3,...,n.$$

where $\mathbf{X} = (x_1, x_2, ..., x_{2n})^T$ - a condition vector.

It is required to find feedback control on a condition vector, such that the border was attracting integral manifold of system (1).

For the decision of this problem we will search for control functions $\mathrm{U}_2(\mathbf{X})$, $\mathrm{U}_{2i}(\mathbf{X})$, $i = 2,3,...,n$, in linear space of smooth polynomials $M_l \in L_R[M_2, M_{1,2},...]$, $\deg(M_l) = 2m+1$, spanning on generators $M_2 = x_2$, $M_{1,2} = x_1^2 x_2$, $M_{2,2} = x_2^3$, $M_{j,2} = x_j^{2m} x_2$, $M_{2i} = x_{2i}$, $M_{1,2i} = x_1^2 x_{2i}$, $M_{2,2i} = x_2^2 x_{2i}$, $M_{j,2i} = x_j^{2m} x_{2i}$, $j = 3,4,...,2n,$ over a field of real numbers R, in a following kind:

$$\mathrm{U}_2(\mathbf{X}) = \beta_1 x_2 + \beta_{1,2} x_1^2 x_2 + \beta_{2,2} x_2^3 + \sum_{j=3}^{2n} \beta_{j,2} x_j^{2m} x_2,$$

$$\mathrm{U}_{2i}(\mathbf{X}) = \beta_{2i-1} x_{2i} + \beta_{1,2i} x_1^2 x_{2i} + \beta_{2,2i} x_2^2 x_{2i} + \sum_{j=3}^{2n} \beta_{j,2i} x_j^{2m} x_{2i}.$$

Synthesis of integral surfaces of Lame and stabilization of movements in their vicinities

Let's enter internal and external semi-vicinities $\partial D^{2n}: \mathrm{B}_\delta^-(\partial D^{2n}), \mathrm{B}_\delta^+(\partial D^{2n})$,

i.e. $\mathrm{B}_\delta^-(\partial D^{2n}) = \mathrm{Int}(D^{2n}) \setminus \{\mathbf{0}\}$, $\mathrm{B}_\delta^+(\partial D^{2n})$,

$\delta > 0$, where $\mathrm{Int}(D^{2n})$ – an interior of domain D^{2n}, i.e.,

$$B_\delta^-(\partial D^{2n}) = \left\{ (X) \in R^{2n} \middle| 0 < \frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2} + \sum_{i=3}^{2n} \frac{x_i^{2m}}{a_i^{2m}} \le 1 \right\}$$

$$B_\delta^+(\partial D^{2n}) = \left\{ (X) \in R^{2n} \middle| 1 < \frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2} + \sum_{i=3}^{2n} \frac{x_i^{2m}}{a_i^{2m}} < 1 + \delta \right\}$$

Theorem (about attraction ∂D^{2n}). *1. That border ∂D^{2n} was integral manifold of system (1) enough performan1ce of following condition on coefficients of control functions:*

$$\begin{cases} \alpha_2 = 2a_2^{-2},\ \alpha_1 = 2a_1^{-2}, \beta_1 = \pm 1, \beta_{1,2} = \mp a_1^{-2}, \\ \beta_{2,2} = \mp a_2^{-2}, \beta_{j,2} = \mp a_j^{-2m}, \alpha_{2i} = 2m a_{2i}^{-2m}, \\ \alpha_{2i-1} = 2m a_{2i-1}^{-2m}, \beta_{2i-1} = \pm 1, \beta_{1,2i} = \mp a_1^{-2}, \\ \beta_{2,2i} = \mp a_2^{-2}, \beta_{j,2i} = \mp a_j^{-2m}, \\ \text{где } i = 2,3,...,n,\ j = 3,4,..2n. \end{cases}$$

2. If thus coefficients of controls obey $\beta_1 = 1$, $\beta_{1,2} = -a_1^{-2}$, $\beta_{2,2} = -a_2^{-2}$, $\beta_{j,2} = -a_j^{-2m}$, $\beta_{2i-1} = 1$, $\beta_{1,2i} = -a_1^{-2}$, $\beta_{2,2i} = -a_2^{-2}$, $\beta_{j,2i} = -a_j^{-2m}$, *border ∂D^{2n} is asymptotically attracting for trajectories, with the entry conditions defined on set* $\mathrm{B}_\delta^-(\partial D^{2n}) \cup \mathrm{B}_\delta^+(\partial D^{2n})$, *i.e., domain* $\mathrm{B}_\delta^-(\partial D^{2n}) \cup \mathrm{B}_\delta^+(\partial D^{2n})$ *is domain of asymptotic stability of integral manifold ∂D^{2n}.*

Consequence. *Let **conditions 2 of theorems** are satisfied**,** then interior $\mathrm{Int}(D^{2n})$ – invariant manifold of system (1).*

In particular, from a theorem consequence follows, that decisions of problem Cauchy, with the initial conditions defined on an interior will be bounded at $t \in (-\infty, +\infty)$.

Proof. 1**. *Invariance of border* ∂D^{2n}**. We will enter into consideration the following function

$$\mathrm{F}(\mathbf{X}) = \frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2} + \sum_{i=3}^{2n} \frac{x_i^{2m}}{a_i^{2m}}.$$

Let's notice, that the entered function has a constant sign as on an interior, its limited surface of level, and in some δ – a layer, adjoining border from the outside. We will calculate a total derivative of function $\mathrm{F}(\mathbf{X})$ on movements of system (1) taking into account conditions 1of **theorem** on coefficients. Then we will receive a following condition:

$$\sum_{i=1}^{n} \left(\dot{x}_{2i-1} \frac{\partial \mathrm{F}(\mathbf{X})}{\partial x_{2i-1}} + \dot{x}_{2i} \frac{\partial \mathrm{F}(\mathbf{X})}{\partial x_{2i}} \right) = \mathrm{P}(\mathbf{X})(1 - \mathrm{F}(\mathbf{X})),$$

or

$$\pm\alpha_2 x_2 \frac{\partial \mathrm{F}(X)}{\partial x_1} + \left(\mp\alpha_1 x_1 + \mathrm{U}_2^{\mp}(\mathbf{X})\right)\frac{\partial \mathrm{F}(X)}{\partial x_2} +$$

$$\sum_{i=2}^{n}\left(\pm\alpha_{2i} x_{2i}^{2m-1}\frac{\partial \mathrm{F}(X)}{\partial x_{2i-1}} +\right.$$

$$\left.\left(\mp\alpha_{2i-1} x_{2i-1}^{2m-1} + U_{2i}^{\mp}(\mathbf{X})\right)\frac{\partial \mathrm{F}(\mathbf{X})}{\partial x_{2i}}\right)$$

$$= \mathrm{P}(\mathbf{X})\left(1-\mathrm{F}(\mathbf{X})\right),$$

where $\mathrm{P}(\mathbf{X}) = \pm 2\left(\frac{x_2^2}{a_2^2} + m\sum_{i=2}^{n}\frac{x_{2i}^{2m}}{a_{2i}^{2m}}\right)$, signs "$+$" or "$-$" take before coefficients of control functions, depending on that "$-$" or "$+$" is available before members of first degree x_{2i-1}, x_{2i} of control.

The received condition is a condition of invariance of the manifold set by equation $1-\mathrm{F}(\mathbf{X}) = 0$. In our case this manifold is border

$$\partial D^{2n} \equiv \left\{(X) \in R^{2n} \left| 1 - \frac{x_1^2}{a_1^2} - \frac{x_2^2}{a_2^2} - \sum_{i=3}^{2n}\frac{x_i^{2m}}{a_i^{2m}} = 0\right.\right\}$$

of manifold D^{2n}, i.e., the invariance condition is carried out only for one of surfaces of level of function $\mathrm{F}(\mathbf{X})$, for ∂D^{2n}.

Thus, border ∂D^{2n} is integral manifold of system.

Proof. 2**. *Asymptotic stability of* ∂D^{2n}.** Let feedback controls on a vector - to a condition look like:

$$\begin{cases} \mathrm{U}_2^{-}(\mathbf{X}) = \beta_1 x_2 + \beta_{1,2} x_1^2 x_2 + \beta_{2,2} x_2^3 + \sum_{j=3}^{2n}\beta_{j,2} x_j^{2m} x_2, \\ \mathrm{U}_{2i}^{-}(\mathbf{X}) = \beta_{2i-1} x_{2i} + \beta_{1,2i} x_1^2 x_{2i} + \beta_{2,2i} x_2^2 x_{2i} + \\ +\sum_{j=3}^{2n}\beta_{j,2i} x_j^{2m} x_{2i}, \\ npu \quad i = 2,3,\dots,2n, \end{cases}$$

where $\beta_1 = 1,\ \beta_{1,2} = -a_1^{-2},\ \ \beta_{2,2} = -a_2^{-2}$,

$\beta_{j,2} = -a_j^{-2m},\ \beta_{2i-1} = 1\ ,\beta_{1,2i} = -a_1^{-2}, \beta_{2,2i} = -a_2^{-2}$,

$\beta_{j,2i} = -a_j^{-2m}$.

Let's consider a total derivative of function on set $\mathrm{B}_\delta^-\left(\partial D^{2n}\right) \bigcup \mathrm{B}_\delta^+\left(\partial D^{2n}\right)$. On interior $\mathrm{B}_\delta^-\left(\partial D^{2n}\right)$ with the pricked out origin of coordinates, the total derivative looks like:

$$\frac{d}{dt}\mathrm{F}(\mathbf{X}) =$$

$$2\left(\frac{x_2^2}{a_2^2} + m\sum_{i=2}^{n}\frac{x_{2i}^{2m}}{a_{2i}^{2m}}\right)\left(1 - \frac{x_1^2}{a_1^2} - \frac{x_2^2}{a_2^2} - \sum_{i=3}^{2n}\frac{x_i^{2m}}{a_i^{2m}}\right),$$

i.e. $\frac{d}{dt}\mathrm{F}(\mathbf{X})-$ it is definitely positive. Owing to what, $\mathrm{F}(\mathbf{X})$ is Lyapunov's function on $\mathrm{B}_\delta^-\left(\partial D^{2n}\right)$. Certain positivity means, that system trajectories on $\mathrm{B}_\delta^-\left(\partial D^{2n}\right)$ will be directed to border from within, crossing surfaces of level functions $\mathrm{F}(\mathbf{X})$ laying on $\mathrm{B}_\delta^-\left(\partial D^{2n}\right)$.

In external δ – a layer, $\frac{d}{dt}\mathrm{F}(\mathbf{X})$ will be definitely negative, i.e., $\mathrm{F}(\mathbf{X})$ also is Lyapunov's function on $\mathrm{B}_\delta^+\left(\partial D^{2n}\right)$, and the trajectories defined in this layer, will be directed, thus, from the outside to border ∂D^{2n}.

Besides, on $\mathrm{B}_\delta^-\left(\partial D^{2n}\right) \bigcup \mathrm{B}_\delta^+\left(\partial D^{2n}\right)$ is not present ω and α – limit points, as well as on surface of Lame α – limit points. According to Zubov's theorem [3] about asymptotic stability of invariant sets taking into account behavior of trajectories on $\mathrm{B}_\delta^-\left(\partial D^{2n}\right) \bigcup \partial D^{2n} \bigcup \mathrm{B}_\delta^+\left(\partial D^{2n}\right)$ we come to conclusion, that set $\mathrm{B}_\delta^-\left(\partial D^{2n}\right) \bigcup \mathrm{B}_\delta^+\left(\partial D^{2n}\right)$ is domain of an attraction for ∂D^{2n}. The theorem is proved.

For the proof of a consequence **of the theorem** it is enough to take advantage of the following statement:

Theorem (Zubov V.I. [3]) *Domain of an attraction of the closed invariant asymptotically stable set is open invariant set.*

Considering, that the attraction domain consists of two non-overlapping sets, and applying the theorem **(Zubov V.I. [3]),** we receive invariance $\mathrm{B}_\delta^-\left(\partial D^{2n}\right)$.

Stabilizing self-excited oscillator and synthesis of orbital stable limit cycles of Lame

Problem statement. It is required to synthesize the controlling generator of self-oscillations, such that in subspace of conditions of an executive part of system the self-oscillatory mode was excited. Geometrically the problem is reduced to synthesis of the feedback control, reducing to emergence of orbital stable limit cycle in subspace of an executive part of system. The given case corresponds to a situation at $i = 2$:

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = -\omega^2 x_1 + \mathrm{U}_{11}(x_1, x_2) + \mathrm{U}_{12}(x_2, x_3, x_4), \\ \dot{x}_3 = x_4^{2m-1}, \\ \dot{x}_4 = -\alpha_3^2 x_3^{2m-1} + \mathrm{U}_{22}(x_3, x_4) + \mathrm{U}_{21}(x_1, x_2, x_4). \end{cases}$$

where U_{11}, U_{22} − required internal feedback controls on a condition vector of the controlling generator of self-oscillations and an controlled subsystem, U_{12}, U_{21} − required feedback controls on a condition vector of all system, providing nonlinear interaction between controlling generator of self-oscillations and an executive part of system. Thus $X_1 \times X_2 \subset R^2$ − subspace of controlling the generator, $X_3 \times X_4 \subset R^2$ - subspace of an executive components.

Let's notice, that standard replacement of variables the system can be copied as follows:

$$\begin{cases} \ddot{\mathrm{y}} + \omega^2 \mathrm{y} + \mathrm{U}_{11}(\mathrm{y}, \dot{\mathrm{y}}) + \mathrm{U}_{12}(\dot{\mathrm{y}}, x_3, x_4) = 0, \\ \dot{x}_3 = x_4^{2m-1}, \\ \dot{x}_4 = -\alpha_3 x_3^{2m-1} + \mathrm{U}_{22}(x_3, x_4) + \mathrm{U}_{21}(y, \dot{y}, x_4). \end{cases}$$

The form of a limit cycle of a controlled subsystem is defined in subspace of movements by function: $\mathrm{F}_2(x_3, x_4) = 1 - \sum_{i=3}^{4} \frac{x_i^4}{a_i^4}$; the required differential equation of the generator of self-oscillations in absence of

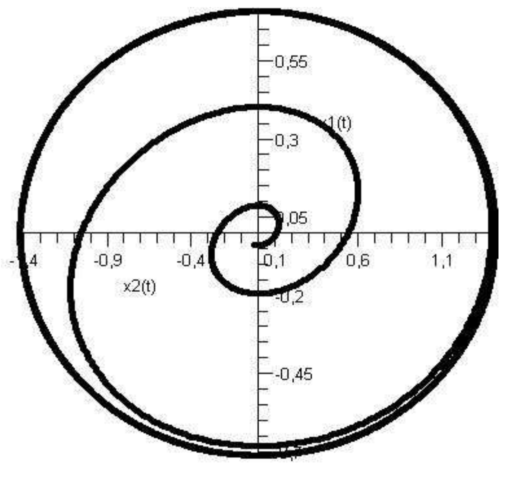

Fig. 2.

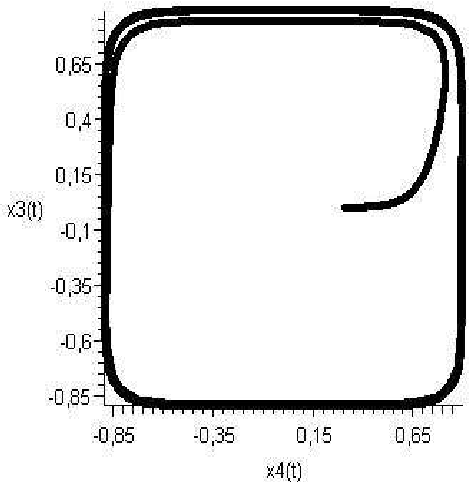

Fig. 3.

interaction with a subsystem of movements looks like: $\ddot{\mathrm{y}} + \omega^2 \mathrm{y} - \beta_1 \dot{\mathrm{y}} - \beta_{1,2} y^2 \dot{y} - \beta_{2,2} \dot{y}^3 = 0$.

Thus, the essential role is played thus by two non - linearity: $y^2 \dot{y}$ - non-linearity of Van der Pol, $\dot{y}^3$ - non-linearity of Rayleigh.

The form of a trajectory of a controlling subsystem is set by function:

$\mathrm{F}_1(x_1, x_2) = 1 - \frac{x_1^2}{a_1^2} - \frac{x_2^2}{a_2^2}$, or in variables $(\mathrm{y}, \dot{\mathrm{y}})$:

$\mathrm{F}_1(y, \dot{y}) = 1 - \frac{\mathrm{y}^2}{a_1^2} - \frac{\dot{\mathrm{y}}^2}{a_2^2}$, where $a_2^2 = \omega_2^2 a_1^2$.

At an exit on a self-oscillatory mode of the controlling generator of a Fig. 2, in space of displacements of system of Lame the orbital stable limit cycle of a Fig. 3 is formed. The offered access allows to control not only geometry of limit cycles and the sizes of the domain limited to these cycles [2].

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