398. New concepts of nonlinear dynamics: complete bifurcation groups, protuberances, unstable periodic infinitiums and rare attractors

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Abstract. A new approach for the global bifurcation analysis for strongly nonlinear dynamical systems, based on the ideas of Poincaré, Birkhoff and Andronov, is proposed. The main idea of the approach is a concept of complete bifurcation groups and periodic branch continuation along stable and unstable solutions, named by the author as a method of complete bifurcation groups (MCBG). The article is widely illustrated using archetypal dynamical systems with one-degree-of-freedom. Among them are: Duffing model (symmetrical, asymmetrical) with one and two potential wells, piecewise-linear systems with one and several potential wells, impact and pendulum systems.

Keywords: dynamical system (DS), bifurcation, complete bifurcation group, method of complete bifurcation group (MCBG), protuberance, unstable periodic infinitium (UPI), rare attractors, catastrophes, strongly nonlinear driven systems, pendulum DS, impact DS, piecewise-linear DS

1. Introduction

The work is devoted to the global analysis of the models of strongly nonlinear dynamical systems by direct numerical and/or analytical methods. The studied nonlinear ODE or discrete models may be with different positional or dissipative terms, such as polynomial, piecewise-linear, piecewise-smooth and others. In distinction to traditional methods and approaches, a new so-called method of complete bifurcation groups (MCBG) is proposed and illustrated. The main idea of the approach is a concept of complete bifurcation groups and periodic branch continuation along stable and unstable solutions, named by the author as a method of complete bifurcation groups (MCBG), based on the ideas of Poincaré, Birkhoff, Andronov and others [1-22]. The main features and advantages of this method are illustrated by archetypical strongly nonlinear driven systems with one degree of freedom [41-52]. Using the method for the systems with several degrees of freedom see in [41,47,51,52]. It is shown that the MCBG allows to find some important unknown regular and chaotic attractors and new bifurcation groups in all nonlinear models under consideration.

New bifurcation sub-groups, received by the MCBG, are so-called rare attractors (RA), complex protuberances, unstable periodic infinitiums (UPI) allow to do complete bifurcation analysis for the nonlinear models. It is shown that UPI bifurcation groups formed in the model chaotic behaviour such as usual chaotic or rare chaotic attractors, or chaotic transients. The complete bifurcation analysis includes stability calculation for each solution and analysis of the kinds of bifurcation points.

The main feature of the MCBG is that it uses nT-branches continuation without their break in bifurcation points and connected with protuberances born from some bifurcation points by period doubling. So unstable periodic solutions, during branch continuation in single parameter space, are corner-stone meaning in the MCBG.

In the paper we also discuss some new qualitative properties of nonlinear dynamical systems received by the method of complete bifurcation groups.

2. Analytical approximate methods and direct "exact" methods. Poincaré point mapping as a main representation of steady-state solutions and transient processes

Usual analytical methods, such as e.g. the van der Pol method, the asymptotic Krylov-Bogolybov method, the averaging methods and the method of harmonic balances not allow to receive all types of steady-state solutions in the strongly nonlinear dynamical systems, systems with discontinuous nonlinearities [23-29] and impact systems [30-35]. Many stationary solutions are complex enough and it is very difficult to guess their form. Fourier presentation, to our mind, is not adequate for nonlinear dynamical systems. The best presentation is Poincaré point mapping, especially for complete bifurcation analysis.

3. Complete and incomplete bifurcation groups and bifurcation diagrams

Bifurcation diagrams (state – parameter) are one of the mostly used in geometric presentation of the qualitative and quantitative behaviour of the nonlinear dynamical systems (Figs. 1,2). But practically all traditional bifurcation diagrams in scientific literature, including received by direct numerical methods, are incomplete: they contain only stable solutions and ignore all unstable ones [12,25-27,36-38 and others]. It will be shown how, based on Poincaré ideas, to build complete bifurcation diagrams. It allows finding new nonlinear effects in the system.

4. Unstable periodic infinitiums (UPI) and the birth of chaos in nonlinear dynamical systems

Several important but insufficiently learned elementary and complex typical bifurcation groups are unstable *periodic infinitiums (UPI)*, rare attractors (RA), protuberances, bifurcation groups with splitting, cascades and chains of subharmonic isles. UPI is a sub bifurcation group based on Poincaré and Birkhoff ideas with infinite unstable periodic solutions nT, 2nT, 4nT, … The existence of the UPI due to the complete cascade of nT-period doubling and the crisis is a necessary part of the typical bifurcation group with chaotic behaviour. It is a well-known fact that the presence of the UPI characterizes the parameter region with chaotic attractors and/or chaotic transients [1-4,17]. See Figs. 3-7,9-11,13,14,16,17. Now it is known, that the system may have several different UPIs simultaneously, and so the result chaotic behaviour depends from each bifurcation group with UPI in this case.

5. Rare Attractors

The concept of rare attractors (RA), recently proposed by the author with his colleagues [41-52] is used for typical important elementary bifurcation groups, which very often stay imperceptible and unnoticed. The different types of rare attractors arise in nT bifurcation groups after a comparatively long (often very long) *continuation of the unstable nT periodic solution* along the parameter. Rare attractors may be periodic, quasi-periodic or chaotic. They may belong to different types such as tip, dumb-bell, hysteresis, and others. Some known examples of rare attractor are: stable hill-top orbits (periodic or chaotic) near unstable equilibrium positions, stable forced oscillations in the systems with the repellent positional forces, narrow periodic windows in chaotic zones, or vice versa, narrow chaotic windows in periodic zones, so-called fugitive subharmonics [12,39], and unexpected dangerous blow-up attractors, such as rare catastrophic phenomena [15]. New typical bifurcation groups with RA are patterns with almost fully unstable subharmonic isles with RA tips (see Figs. 5-8,10-17). In two parameter plane such isles hare thin chaotic coat with chaotic attractors. Rare attractors exist to our mind in all strongly nonlinear systems including piecewise-smooth and impact dynamical systems.

6. Protuberances, different bifurcation groups and their interaction

Protuberances, as well as UPI and RA, are also important but poorly known parts of nT complete bifurcation groups. Protuberances grow up from the internal bifurcation parameter points (p1, p2), where stable and unstable nT solutions change their stability. Protuberances may extend far from points p1, p2 and their topological structures often are very complex (but complete and very beautiful) even for the most simple oscillators (see Figs. 1-5,10,17). Protuberances may have their own folds, UPI and RA. The theory of protuberances allows predicting and explaining some unexpected phenomena generated from other parameter regions. It is known that several different complete bifurcation groups may exist in the same parameter region. Their *interaction* generates multiplicity and complexity. In a case of interaction, different bifurcation groups involved may be simple (subharmonic nT-isles, nT-peninsula, etc.) or complex with their own protuberances, rare attractors RA, UPI, and chaotic attractors. Strong interactions between different bifurcation groups go to the global bifurcations and lead to the crucial changes of the basins of attraction and structural instability.

7. Subharmonic islands (isles) with UPI and rare attractors. Chaotic coats

Nonlinear driven oscillators has several subharmonic bifurcation groups in the island (isle) form. Such subharmonic islands may have simple form with one stable and one unstable branch, but often they have more complex topology (see for example Figs. 5,7,11,13,14, and 16). Subharmonic islands may have UPI(s) and rare attractors of different types: tip RA, dumb-bell, hysteresis, or kink-shaped. Sometimes there are two different UPI in the same parameter region, formed on each island's side. The are specific patterns with many subharmonic islands with tip rare attractors. Such patterns also have been found in driven oscillators with linear restoring force and nonlinear dissipative force. Subharmonic islands with tip RA may form so-called chaotic coats with narrow cover where rare chaotic subharmonic attractors exist.

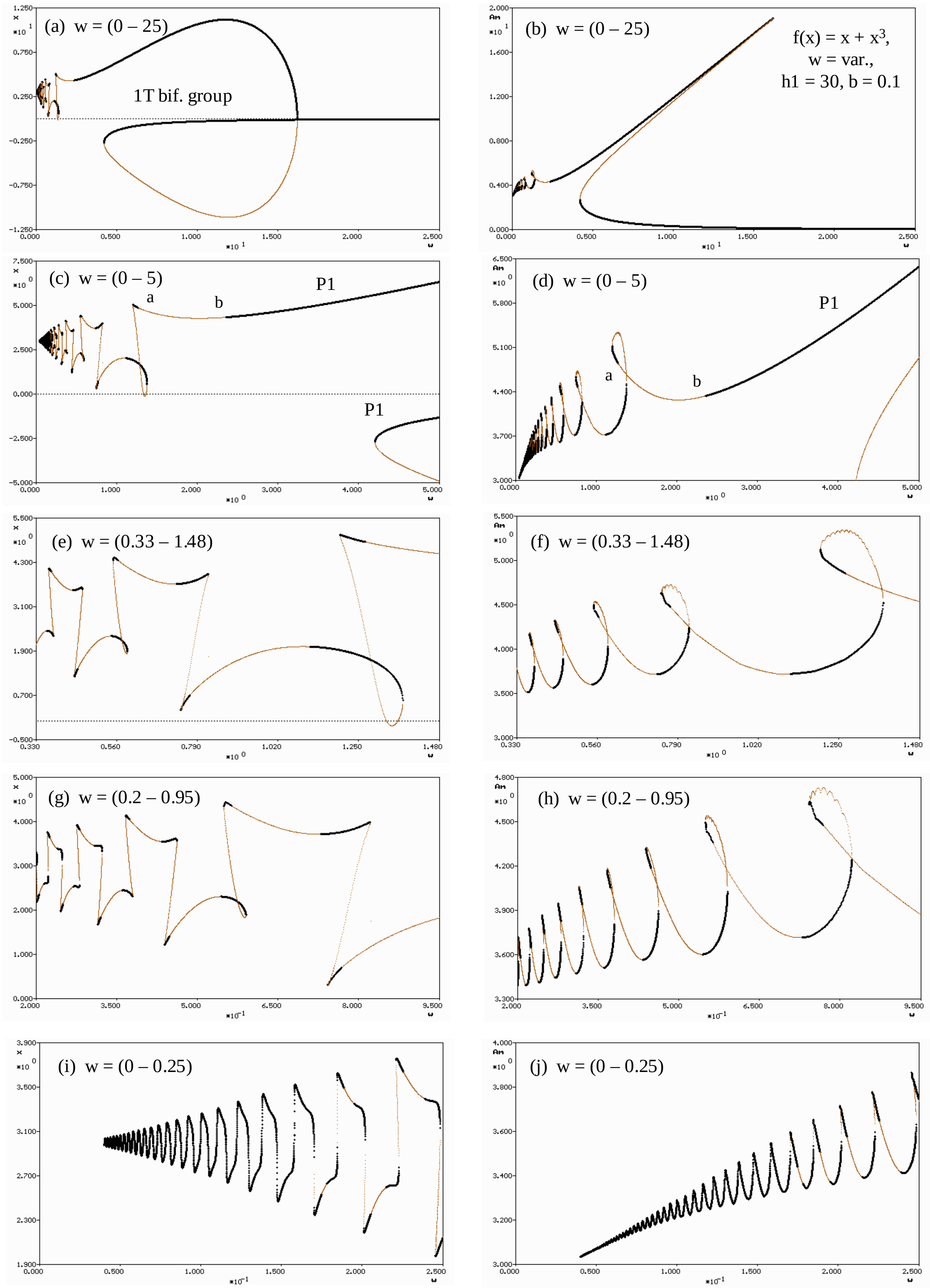

Fig. 1 The Duffing driven damped symmetric system, $f(x) = x + x^3$. Basic parts of the 1T complete bifurcation diagrams with only period-1 (P1) stable (black) and unstable (reddish) symmetric solutions for pre-resonant regions. The coordinates x of the fixed P1 points versus excitation frequency w are shown in the left column. Amplitudes Am versus w are shown in the right column., Parameters: b = 0.1, h1 = 30, w = var.

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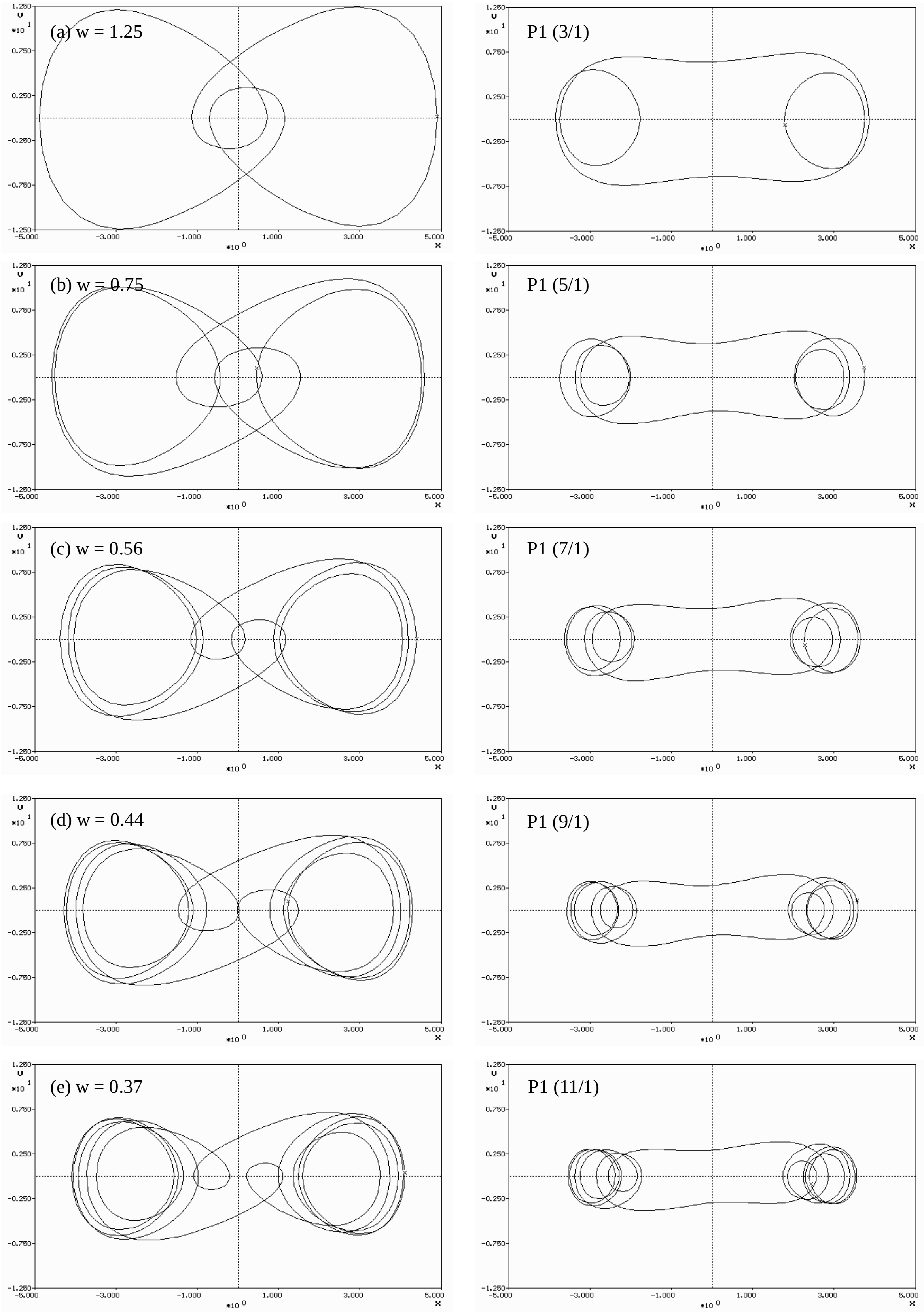


Fig. 2 Duffing driven damped symmetric system with $f(x) = x + x^3$. Period-1 (P1) superharmonical attractors (n/1) from the same 1T bifurcation group for different magnitudes of external frequency w (see Fig. 1). There are two P1 (n/1) odd order (n = 3, 5, …, 11) attractors – resonant (left column) and non-resonant (right column) - for each shown frequency w; so the system has multiplicity of the P1 (n/1) attractors. All shown in the figure regular attractors have folds from the left (see Fig. 1, d – h) and decrease their resonant amplitudes A_m when external frequency is increasing; non-resonant P1 attractors on the contrary increase their amplitudes A_m. Parameters: b = 0.1, $h_1 = 30$, w = var.

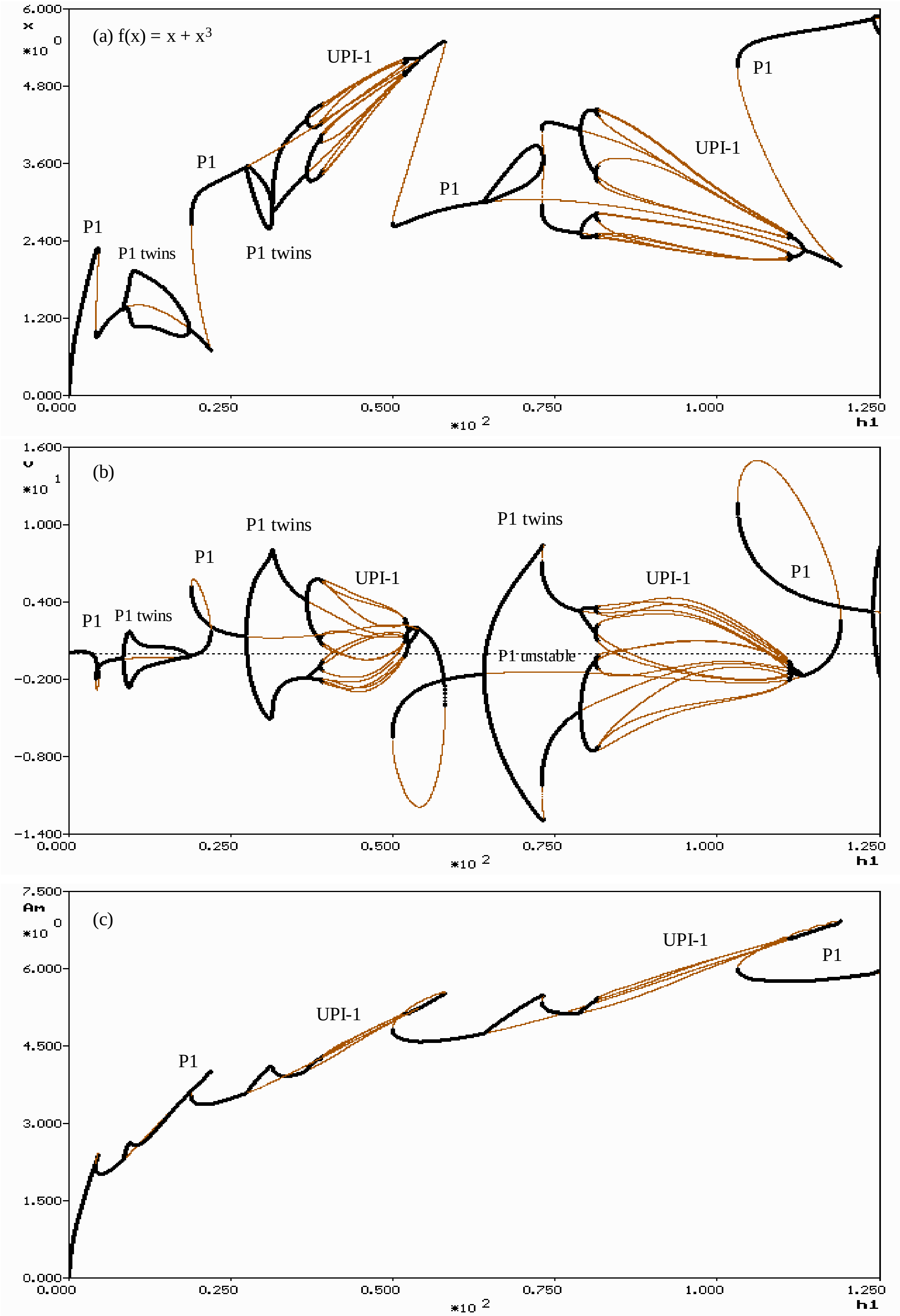
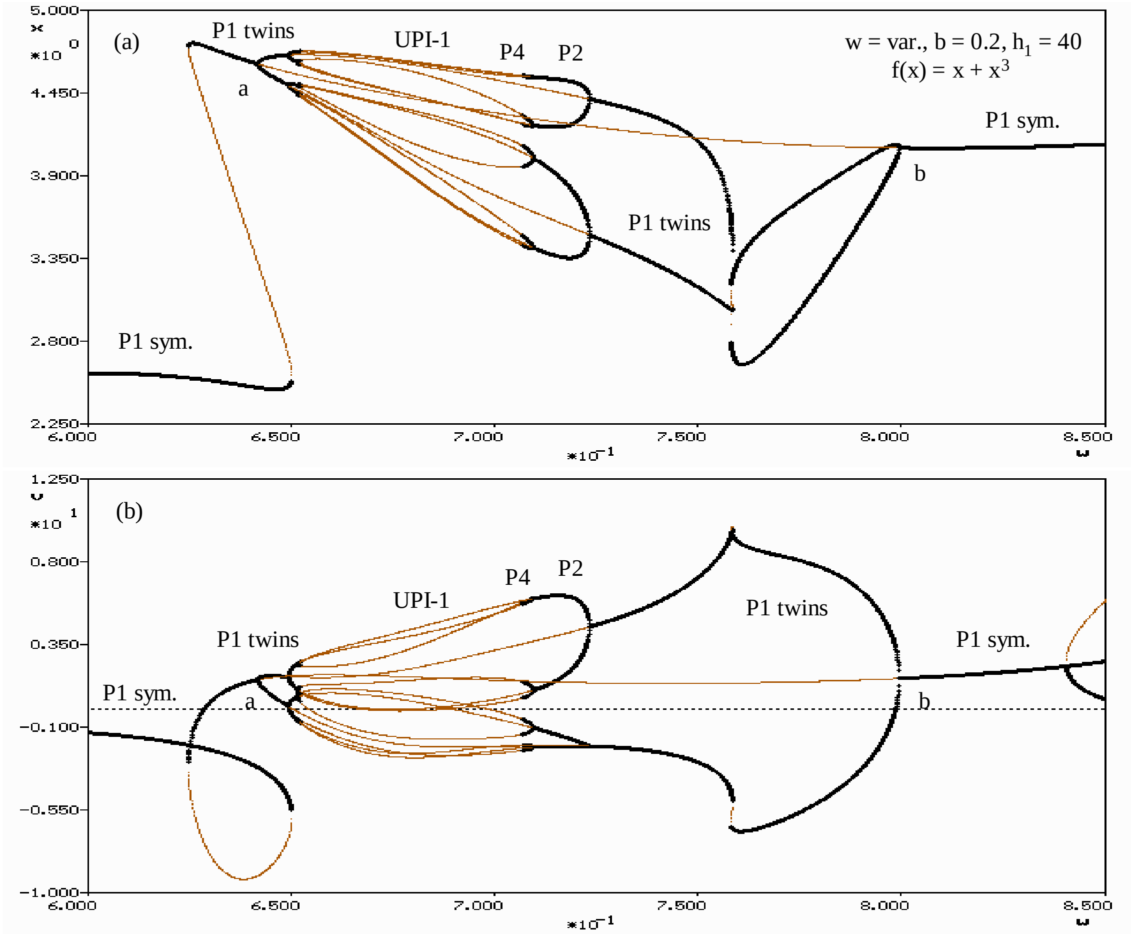
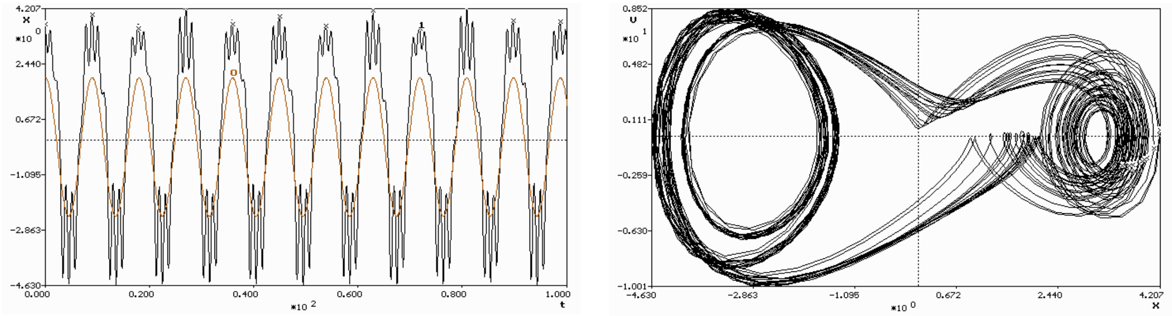
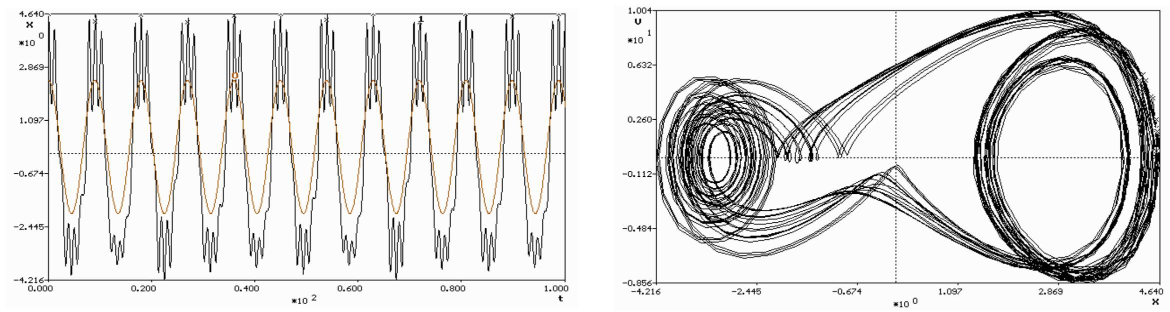


Fig. 3 Duffing harmonically driven damped symmetric oscillator, $f(x) = x + x^3$. Typical 1T bifurcation group with the folds, symmetry breaking bifurcations, period doubling cascades and unstable periodic infinitiums UPI-1. Bifurcation diagrams: state (x, v) of the fixed periodic points and amplitude A_m versus amplitude h_1 of excitation force. Parameters:, damping coefficient b = 0.2, h_1 = var., w = 0.7.





(c) The chaotic left twin attractor. The last periods of 1000T mappings with initial conditions (4.19, 0). Parameters: h_1 = 40, w = 0.7.



(d) The chaotic right twin attractor. The last periods of 1000T mappings with initial conditions (4.57, 0). Parameters: h_1 = 40, w = 0.7.

Fig. 4 Duffing harmonically driven damped symmetric oscillator, $f(x) = x + x^3$. (a), (b) Bifurcation diagrams: state (x, v) of the fixed periodic points vs excitation frequency w. Typical 1T bifurcation group with the symmetry breaking bifurcations, period doubling cascades and unstable periodic infinitium UPI-1 inside the (a, b) protuberance. (c), (d) Chaotic twin attractors from the UPI-1 region. Parameters:, b = 0.2, h1 = 40, w = var.

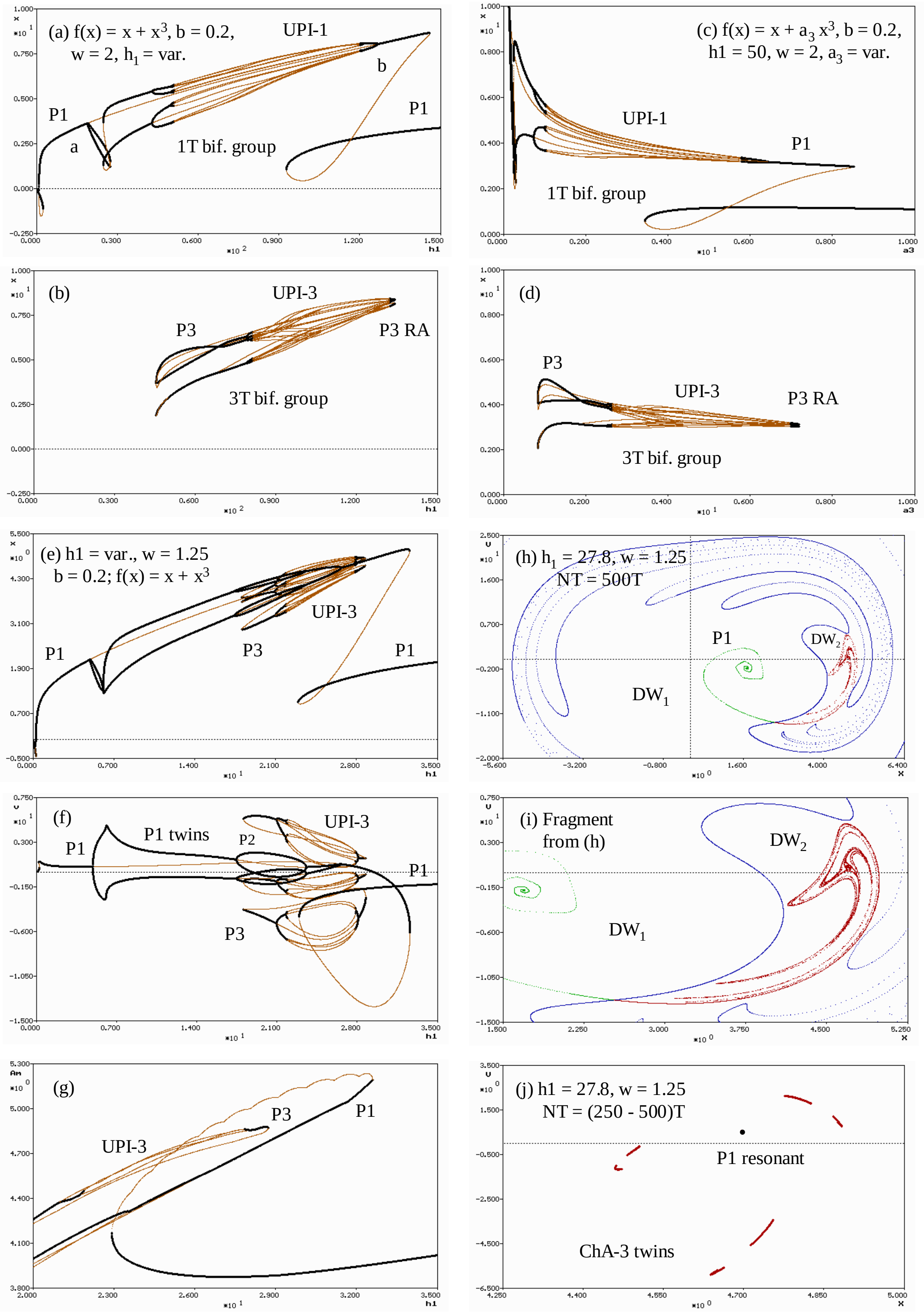


Fig. 5 Duffing harmonically driven damped symmetric oscillator, $f(x) = x + x^3$. Coexisting of two bifurcation groups 1T and 3T with UPI. (a), (b) Bifurcation diagrams of 1T and 3T groups: state x of the fixed periodic points vs amplitude h_1 of excitation; (c), (d) the same for $h_1 = 50$, a_3 = var. (e) – (g) Bifurcation diagrams 1T and 3T with UPI-3. The birth of chaotic twin attractors ChA-3 from UPI-3 in the dynamical well DW_2 are shown on the next figures (h) – (j). Here an important typical nonlinear phenomenon is illustrated: chaotic rare windows. In the shown case there is coexisting of the rare chaotic subharmonic attractors with the amplitudes greater than usual P1 attractor has. See also Fig. 6. with P1 and ChA-3 time histories and phase projections.

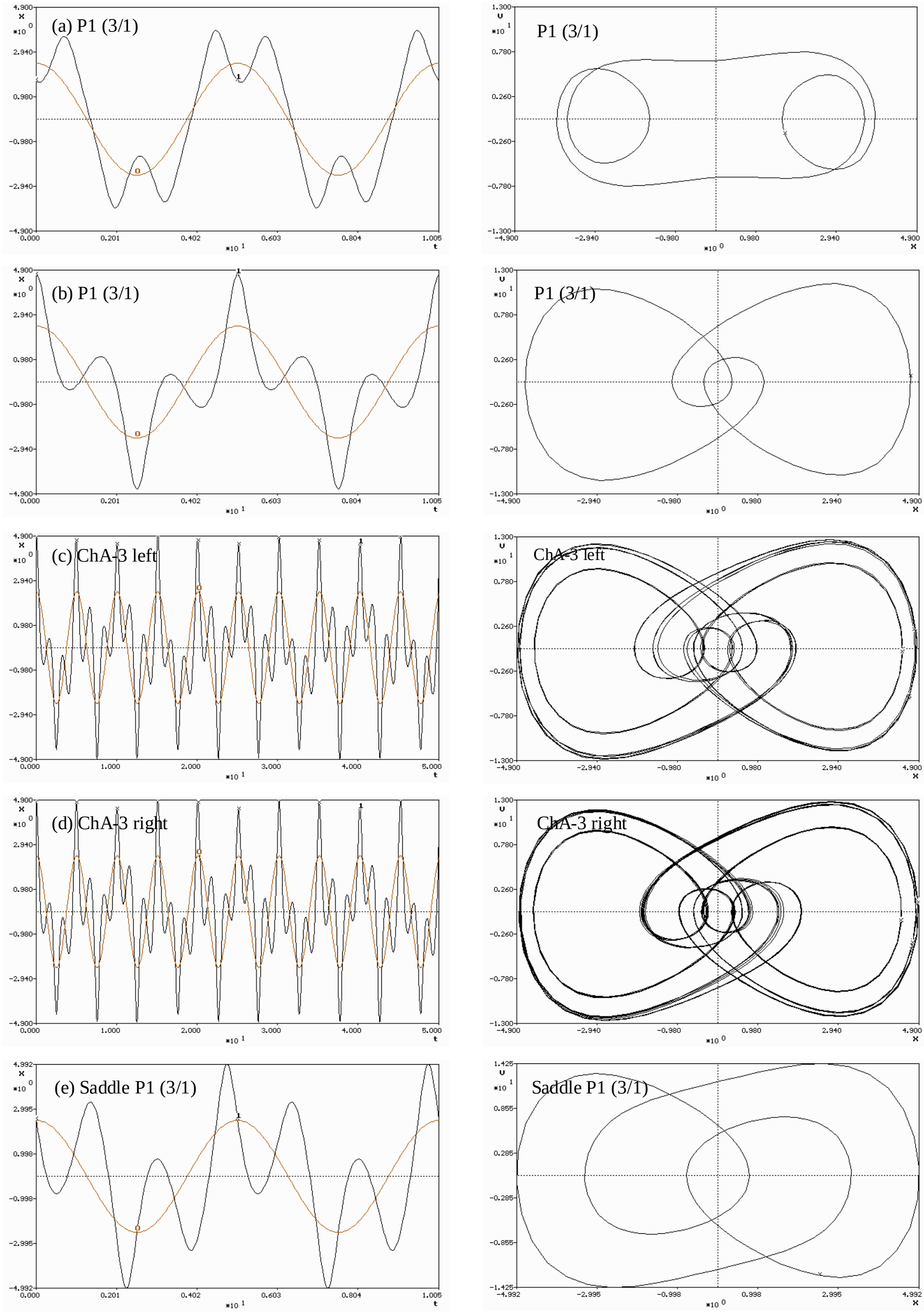


Fig. 6 Duffing harmonically driven damped symmetric system, $f(x)=x+x^3$. The system has two dynamical wells: (a) the left well with P1 (1/3) non-resonant attractor, and (b)-(d) the right complex well with P1 (3/1) resonant and chaotic twin attractors ChA, which have the origin in 3T-island's UPI-3 (see Fig. 5). The boundaries between two dynamical wells DW_1 and DW_2 (see Fig. 5, h, i) have been build from unstable direct saddle (in the figure). The saddle fixed point has coordinates (2.528267, -12.780299) and its multipliers are $\lambda_1 = 3.325$, $\lambda_2 = 0.110$. Parameters of the system: b = 0.2, h1= 27.8, w = 1.25.

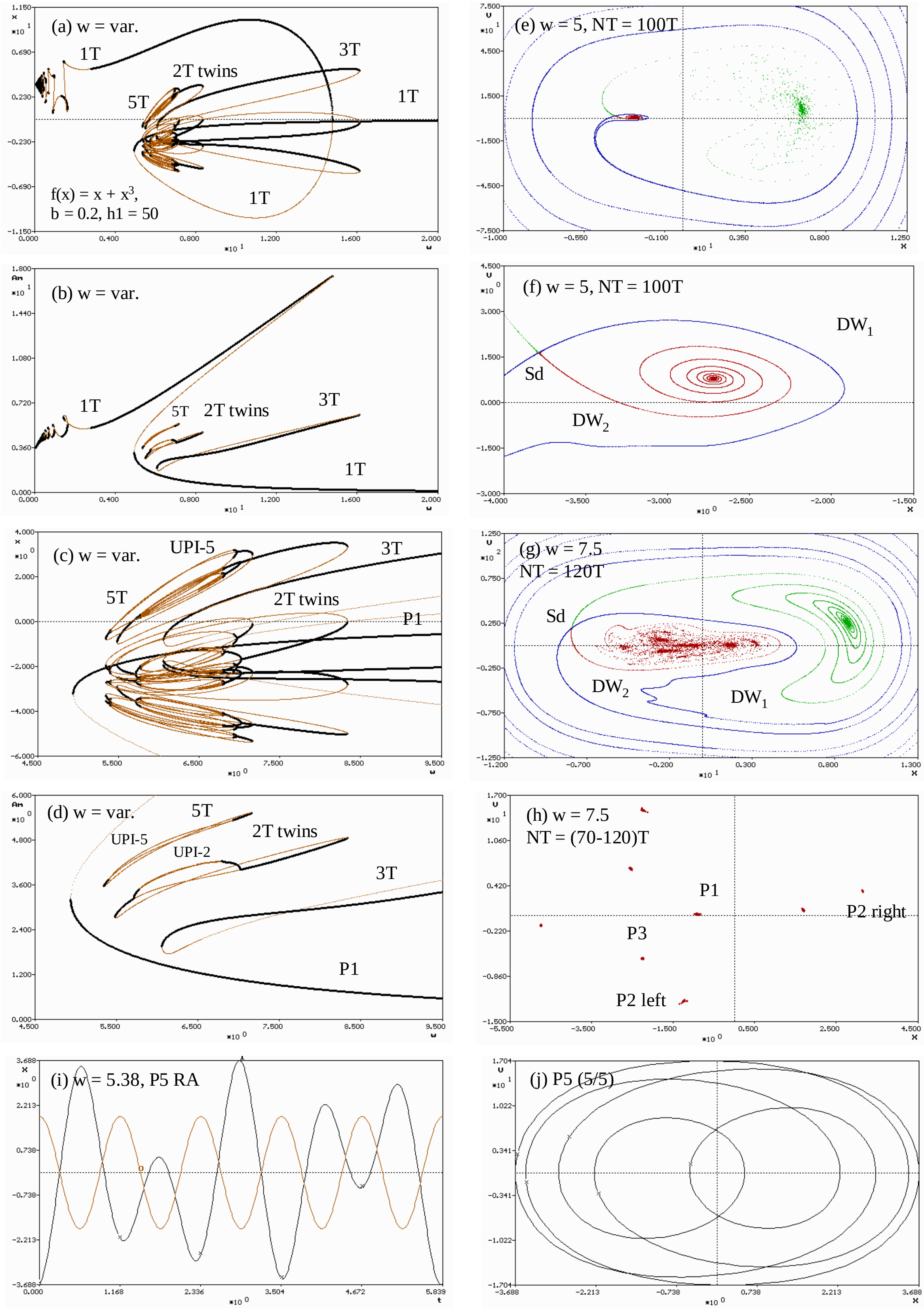


Fig. 7 Symmetric driven damped Duffing system, $f(x) = x + x^3$. The problem of interdependence between the homoclinics and the UPIs of the bifurcation groups. (a) – (d) Bifurcation diagrams of the 1T, 2T twins, 3T, and 5T bifurcation groups. The 2T and 5T subharmonic groups (isles) have UPIs. Rare P5 (5/5) attractor ((i), (j) in the figure) has the fixed point coordinates (- 3.653427, 2.660456), multiplies λ_i = 0.558, α = 74.5°. (e) – (h) Dynamical wells and domains of attraction built by the insets and outset from the P1 saddles. The system hasn't homoclinics for w = 5 and w = 7.5 because there are no UPIs for these values. Parameters of the system: b = 0.2, h1 = 50, w = var.

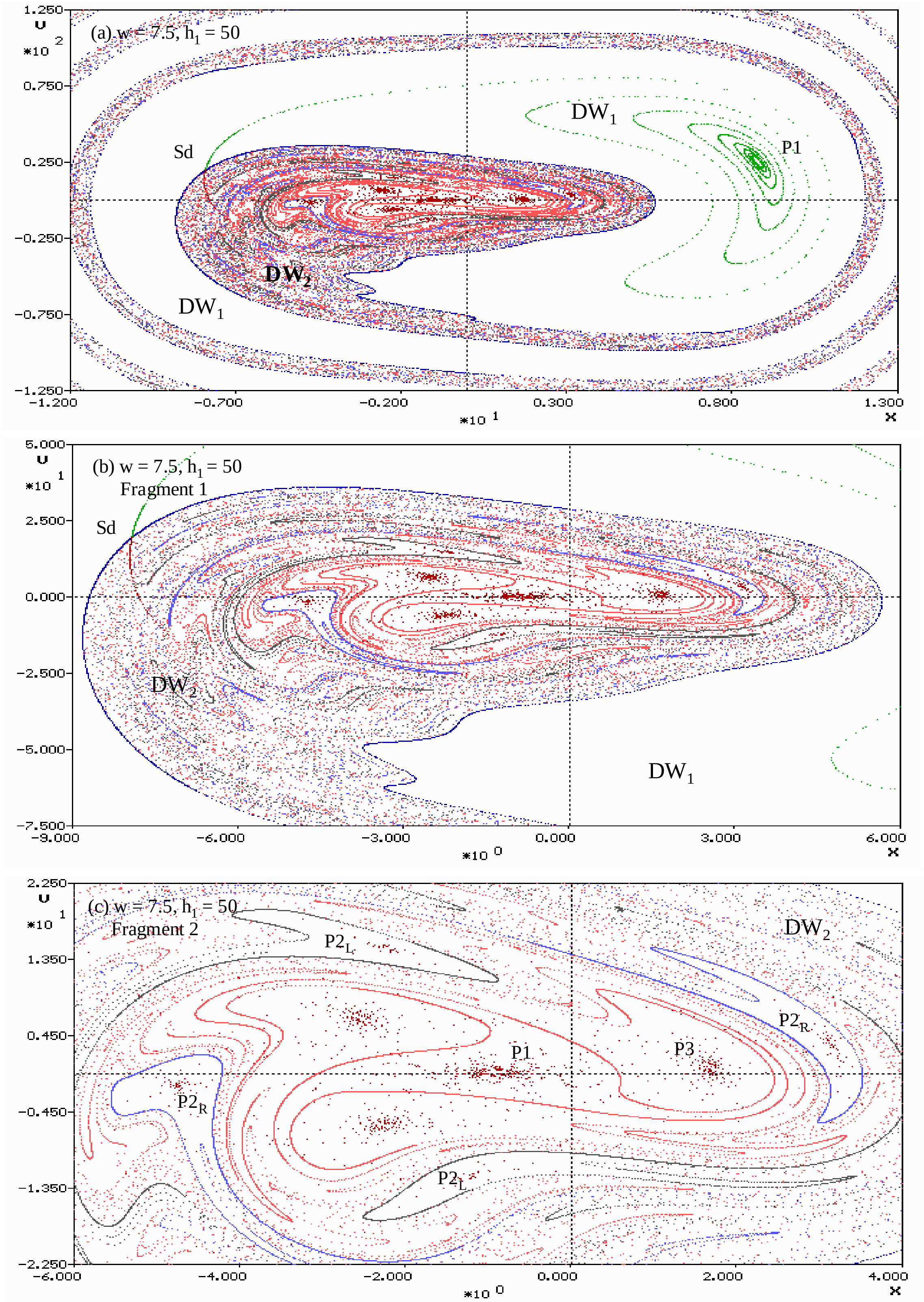


Fig. 8 Duffing symmetric driven system with hardening stiffness and linear dissipation. The problem of interdependence between the homoclinics and the UPIs of the bifurcation groups. Dynamical wells DW_1 and DW_2, and domains of attraction built by the insets and outset from the P1, P2 and P3 saddles on the Poincaré map. The system has not homoclinics because for w = 7.5 there are no UPIs. There are two symmetrical P1 and P3 attractors and P2 twin attractors in the system with their domains of attractions. Parameters: $f(x) = x + x^3$, b = 0.2, h1 = 50, w = 7.5. See also Figure 7, (g), (h) where the dynamical wells and fixed points are plotted.

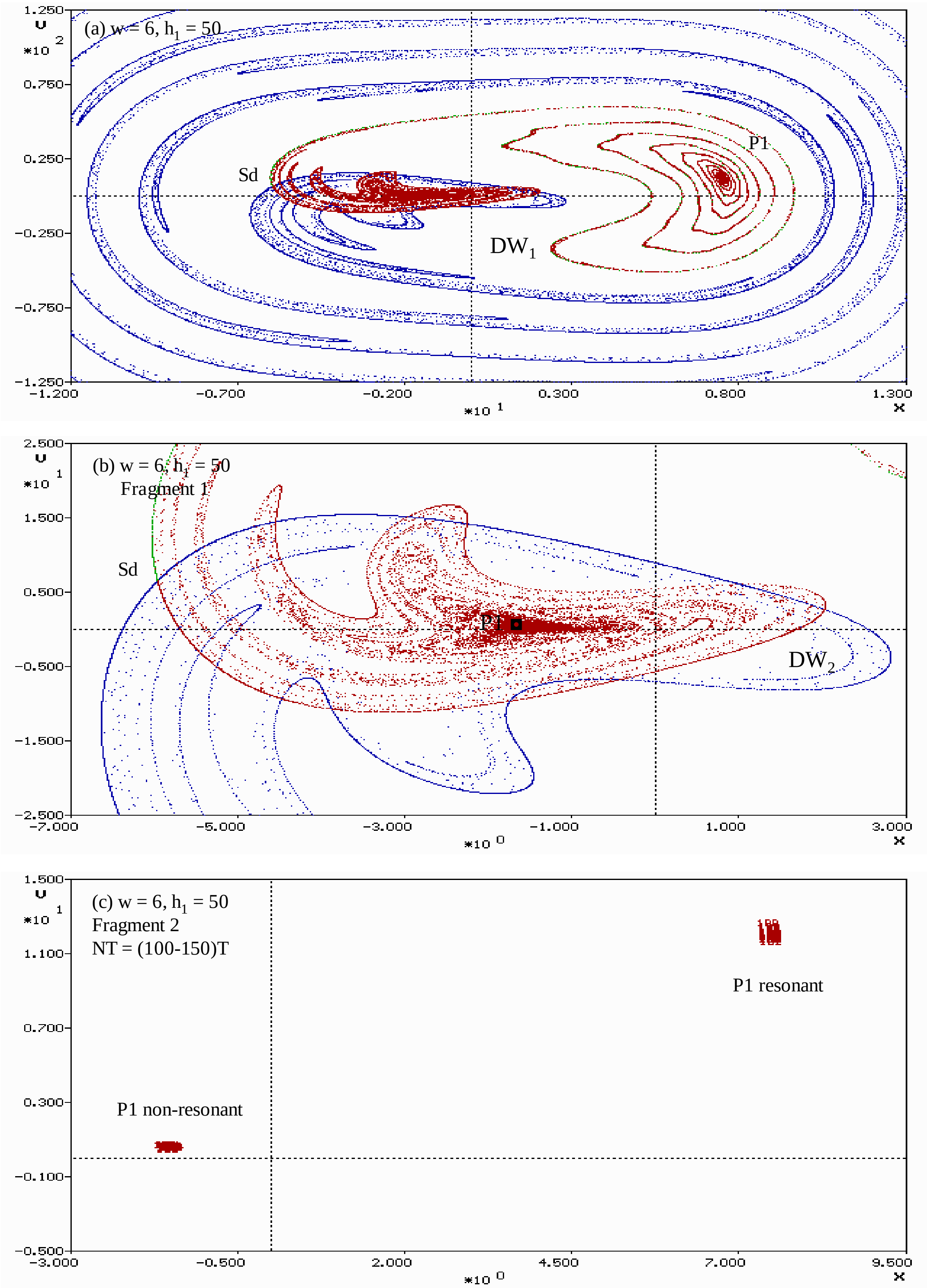


Fig. 9 Duffing symmetric driven system with hardening stiffness, $f(x) = x + x^3$, and linear dissipation. The problem of interdependence between the homoclinics and the UPIs of the bifurcation groups. Dynamical wells and homoclinics built by the insets and outset from the P1 direct saddle (Sd). The system has homoclinics because for w = 6 there are UPI-2 and UPI-5 in the 2T and 5T bifurcation groups (subharmonic 2T and 5T isles). In spite of the existence of homoclinics and UPIs, the system has only two regular attractors: symmetrical P1 resonant and P1 non-resonant ones. Parameters: $b = 0.2$, $h_1 = 50$, $w = 6$.

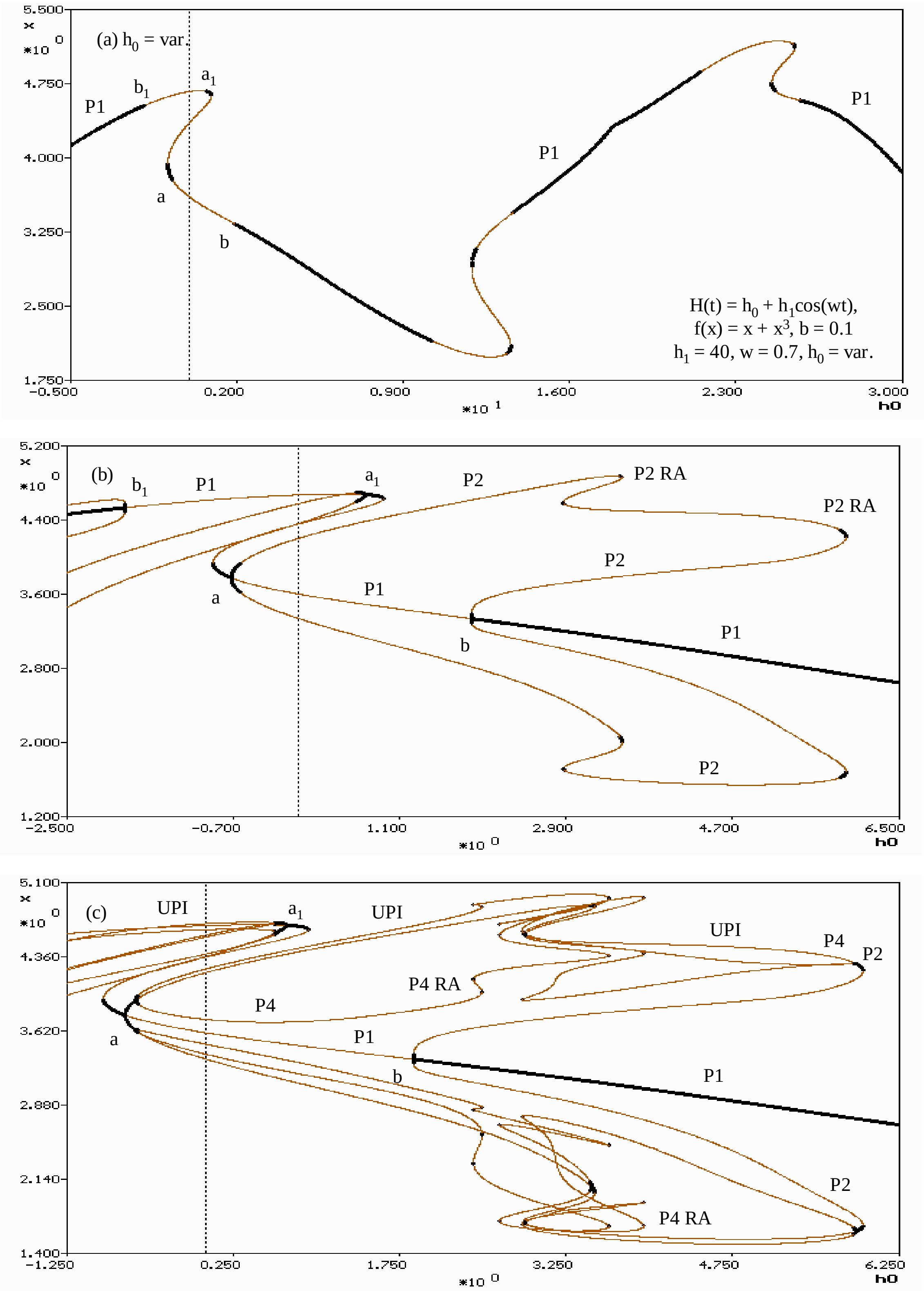


Fig. 10 Complete 1T bifurcation diagrams (h_0 = var.) for the *asymmetric* Duffing driven damped system with stable (black) and unstable (reddish) asymmetric solutions: (a) only period-1 (P1) solutions are shown; (b) with P1 and P2 solutions; (c) with P1, P2 and P4 solutions. The coordinates x of the fixed period-n points are plotted versus the constant part h0 of the excitation harmonic force. The system has P1complex (a-b) protuberance, several P1, P2 and P4. rare attractors (RA) and several unstable periodic infinitiums UPIs. Parameters of the system: $H(wt) = h_0 + h_1\cos(wt)$, $f(x) = x + x^3$, $b = 0.1$, $h_1 = 40$, $w = 0.7$.

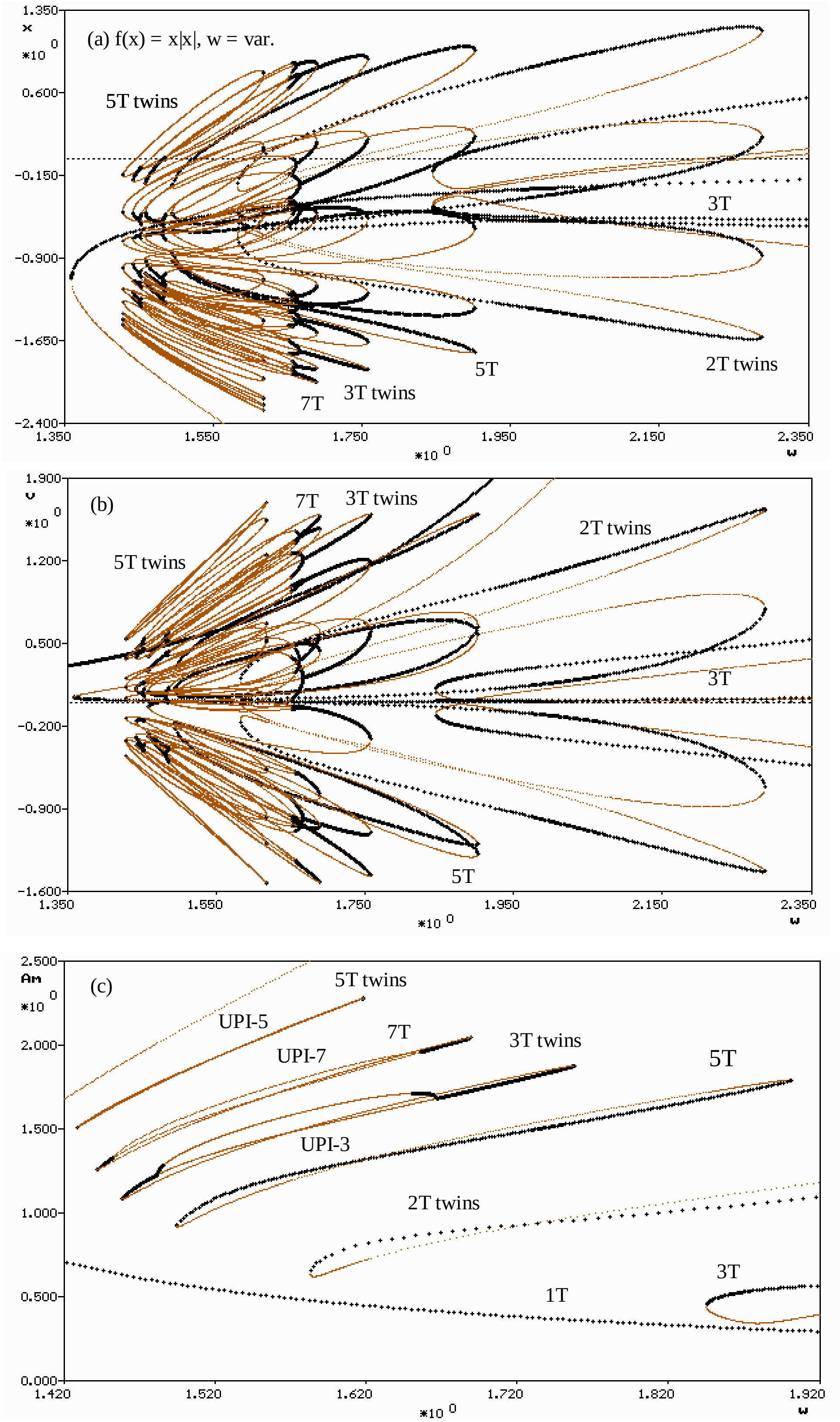


Fig. 11 Quadratic symmetric driven slightly damped system, f(x) = x|x|. Bifurcation diagrams, w = var. In the figure several subharmonic isles are plotted: from left to right: 5T twins with tip rare attractors (RAs), symmetrical 7T with RA, 3T twins (all with periodical unstable infinitiums UPI_n), symmetrical 5T, 2T twins, and symmetrical 3T. Symmetrical 15T isle (1.5374 < w < 1.6899) with unstable infinitiums UPI-15 is not shown in the figure. Parameters: h_1 = 1, b = 0.02.

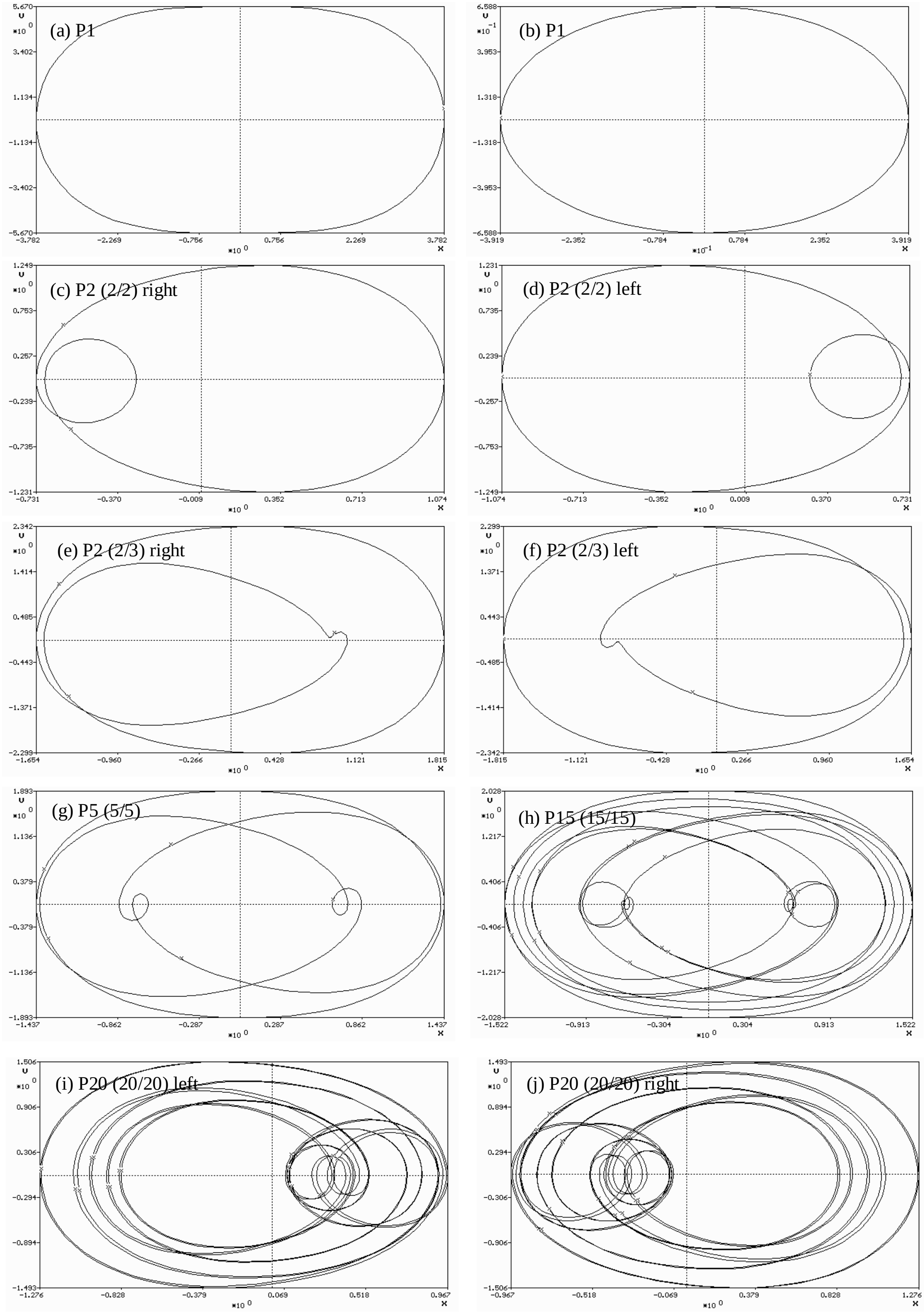


Fig. 12 Quadratic symmetric slightly damped driven system with ten stable periodic solutions (attractors) for the same parameters. There are two main dynamical wells in the system: one simple with P1 resonance stable solution ((a) in the Figure), and another complex dynamic well with nine attractors: (b) non-resonance P1; (c) and (d) two mutually symmetric P2 (2/2) subharmonics (twins); (e) and (f) two mutually symmetric P3 (2/3) subharmonics (twins); g) P5 (5/5) symmetric subharmonic; (h) P15 (15/15) symmetric subharmonic (tripling and n/n subharmonic effects); (i) and (j) two mutually symmetric P20 (20/20) subharmonics. System's parameters: $f(x) = x^2$ sgnx, b = 0.01, h1 = 1, w = 1.7.

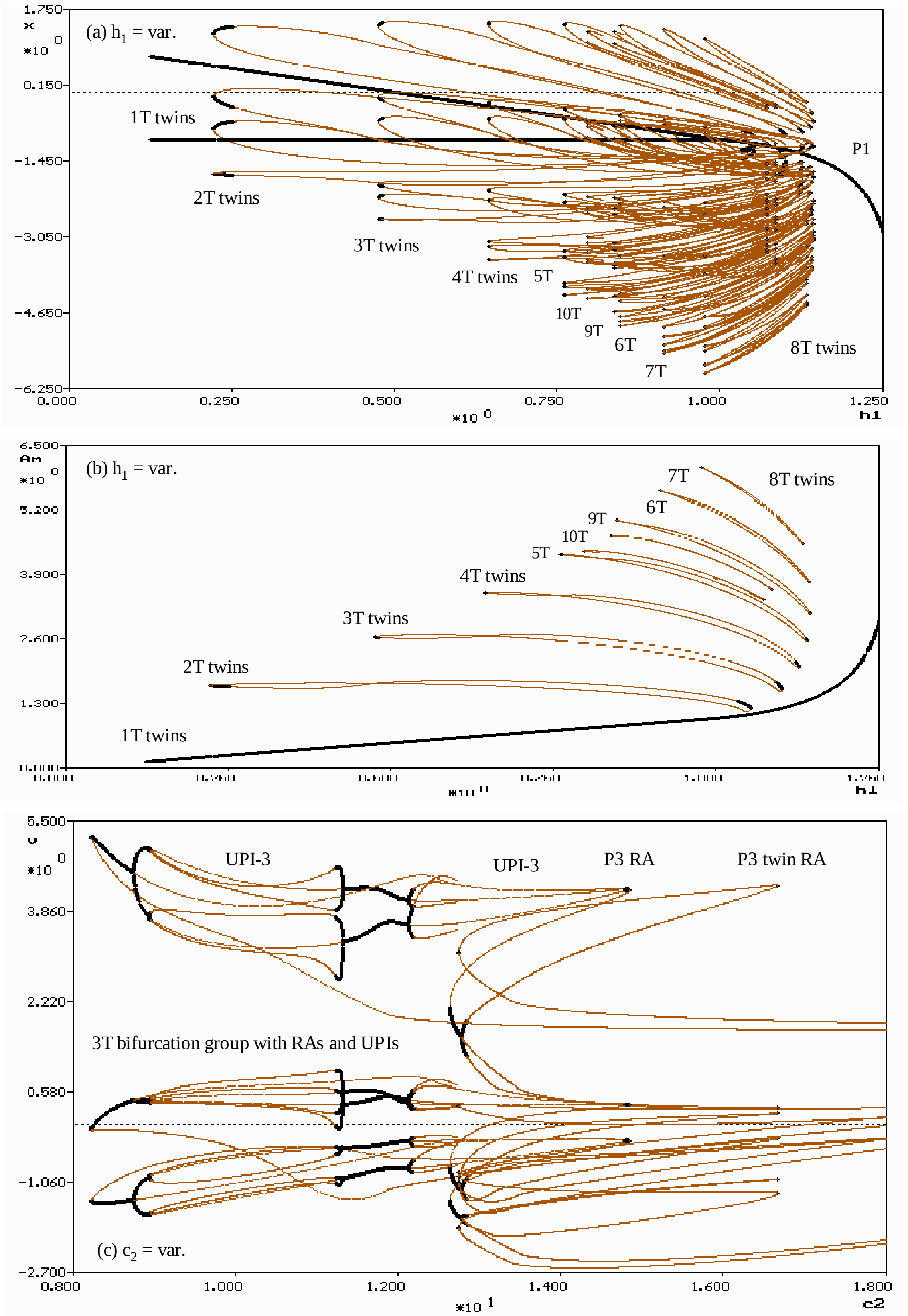


Fig. 13 Trilinear symmetrical system with clearance, linear damping, and harmonic forcing. (a), (b) The system with slight damping, amplitude of excitation h_1 = var. Bifurcation diagrams $S_n(h1)$ with 1T and 2T,…, 9T and 10T bifurcation groups (subharmonic isles). All isles and 1T are twins and they have UPI and tip RA sub-groups with chaotic attractors. Parameters: $c_1 = 0$, $c_2 = 1$, $d = 1$, $b = 0.04$, $w = 1$, $h1$ = var. (c) The system has middle damping value. Bifurcation complex 3T subharmonic group with different rare attractors. New parameters: stiffness c_2 = var., $h_1 = 10$, $b = 0.3$. Black colour is used for stable solutions and reddish for unstable ones.

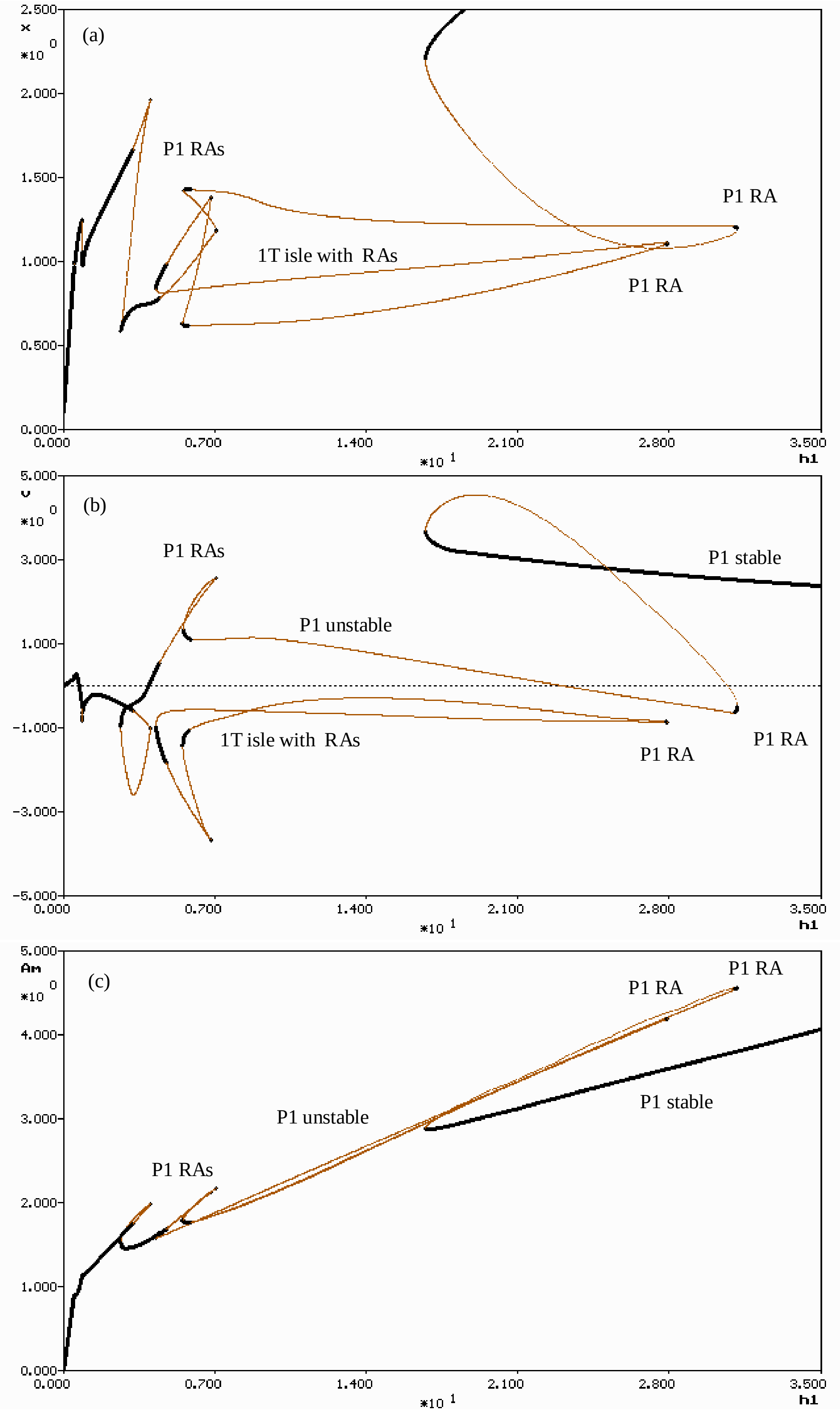


Fig. 14 Trilinear asymmetrical harmonically driven damped system with rare attractors (RA). The system has a slight constant preload h_0 which makes the system asymmetrical one. Due to asymmetric excitation the 1T bifurcation group splits into two groups: the main 1T and 1T isle with additional P1 and chaotic rare attractors. Parameters: c1 = 1, c2 = 16, d = 1, b = 0.2, w = 0.7, h0 = 0 .1, h1 = var.

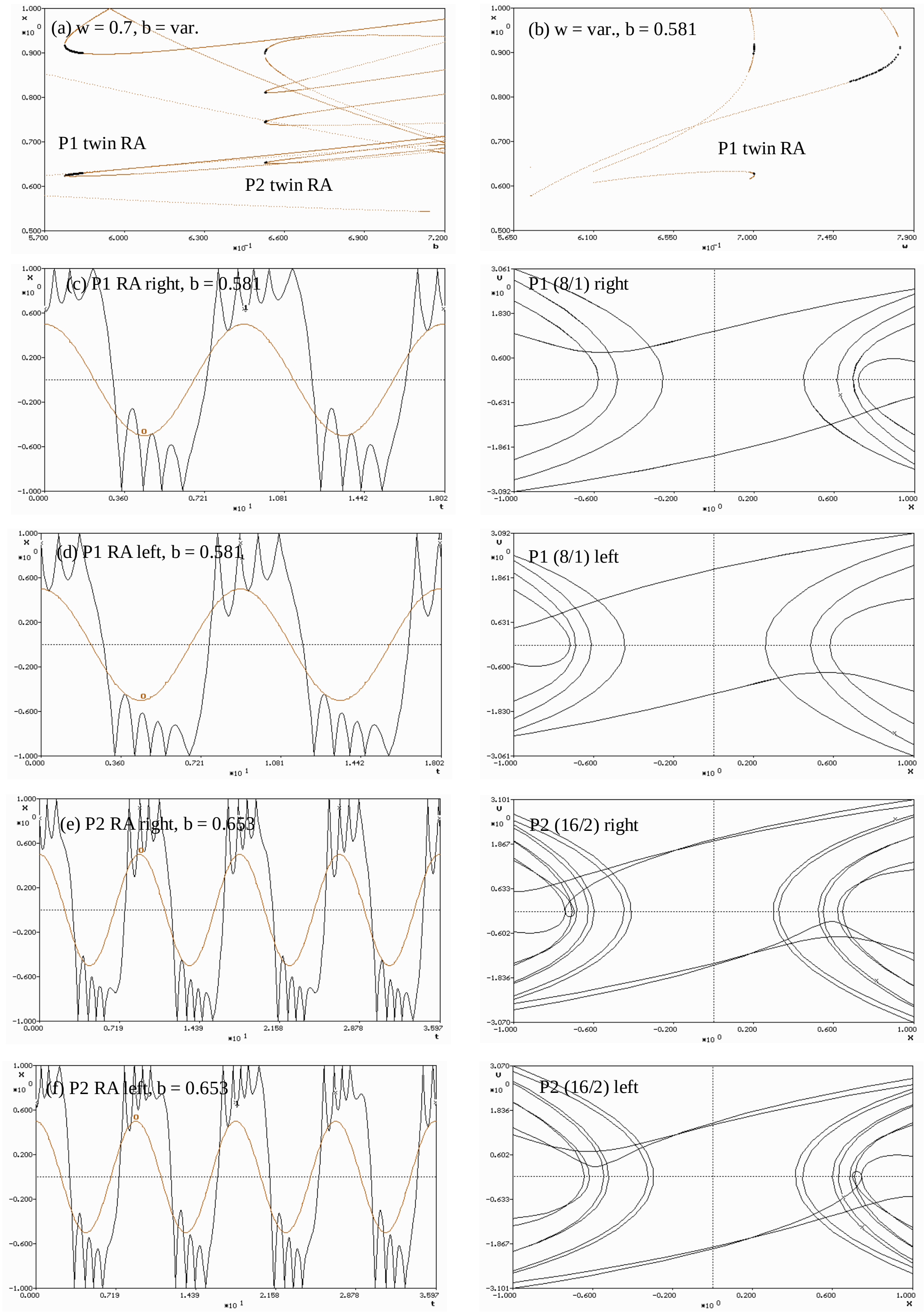


Fig. 15 The birth of asymmetric twin P1 and P2 rare attractors (RA) because of increasing linear damping coefficient b in a vibro-impact symmetrical driven system. (a), (b) Bifurcation diagrams with the rare attractors for w = 0.7, b = var., and w = var., b = 0.581 accordingly; (c), (d) time histories and phase projections of the P1 twin rare attractors, w = 0.7, b = 0.581; (e), (f) time histories and phase projections of the P2 twin rare attractors, w = 0.7, b = 0.653. Parameters of the system: c1 = 1, d1 = 1, d2 = - 1, R1 = R2 = - 0.99, h1 = 7.

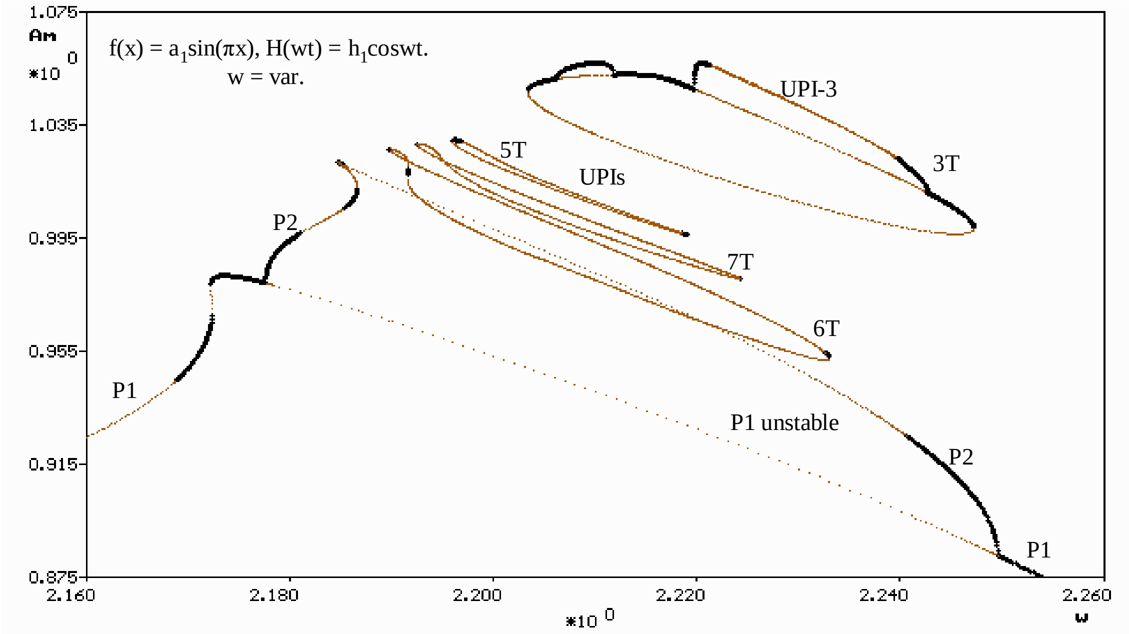


Fig. 16 Harmonically driven strongly damped pendulum system. Bifurcation diagrams $A_m(w)$ with 1T complex bifurcation group and several twin subharmonic isles (3T, 5T, 6T, and 7T) with rare attractors and UPIs in the each isle. Parameters: b = 0.7, a_1 = - 1, h_1 = 4, w = var.

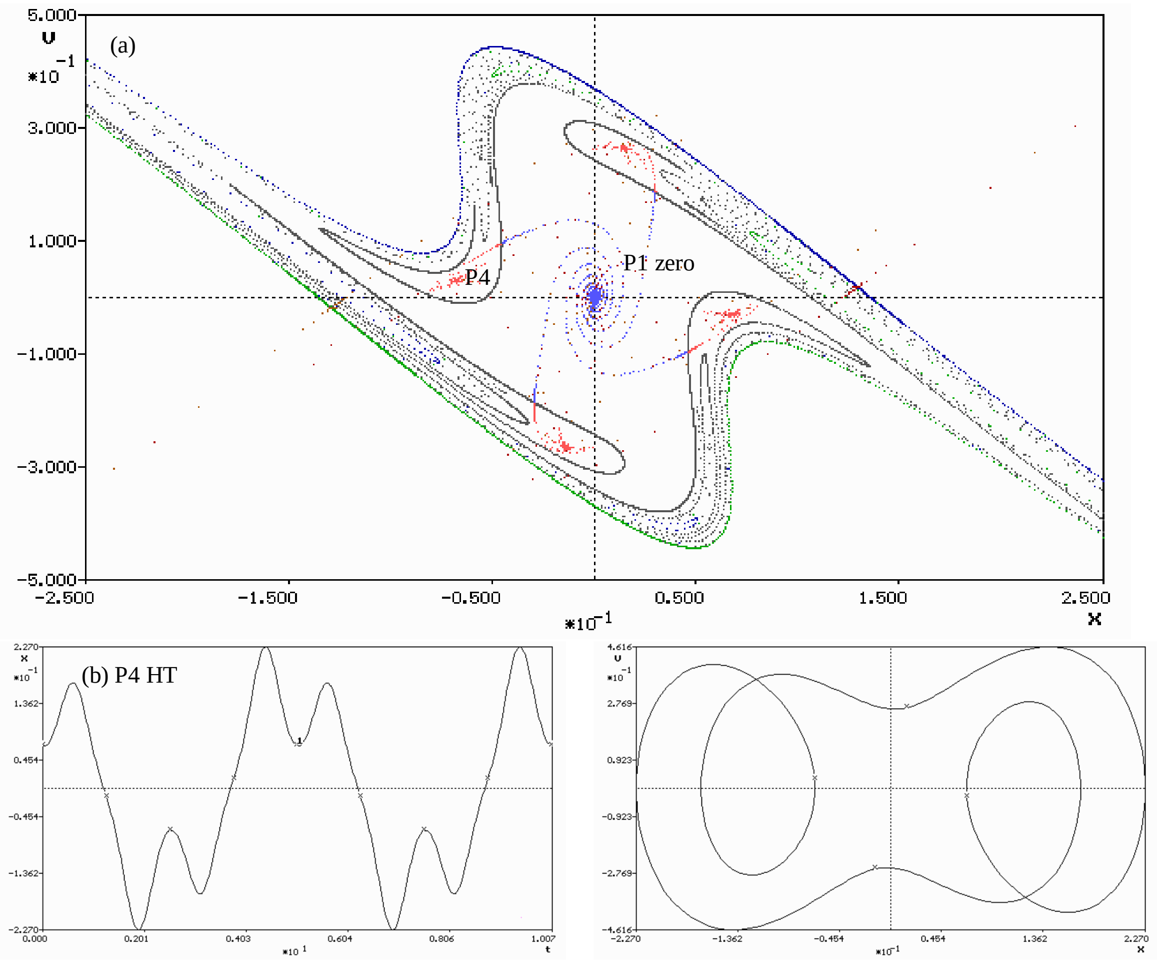


Fig. 17 Pendulum parametric system x..+ bx. - (h_0 + hcoswt) sin(πx) = 0. (a) The cores of the domains of attraction of the hilltop zero P1 and hilltop subharmonic non-zero P4 stable solutions; (b) time history and phase projection of the P4 hilltop attractor. The system has subharmonic n = 2 stable rotations as well. Parameters: b = 0.2, h_0 = 1, h = 5, w = 5.

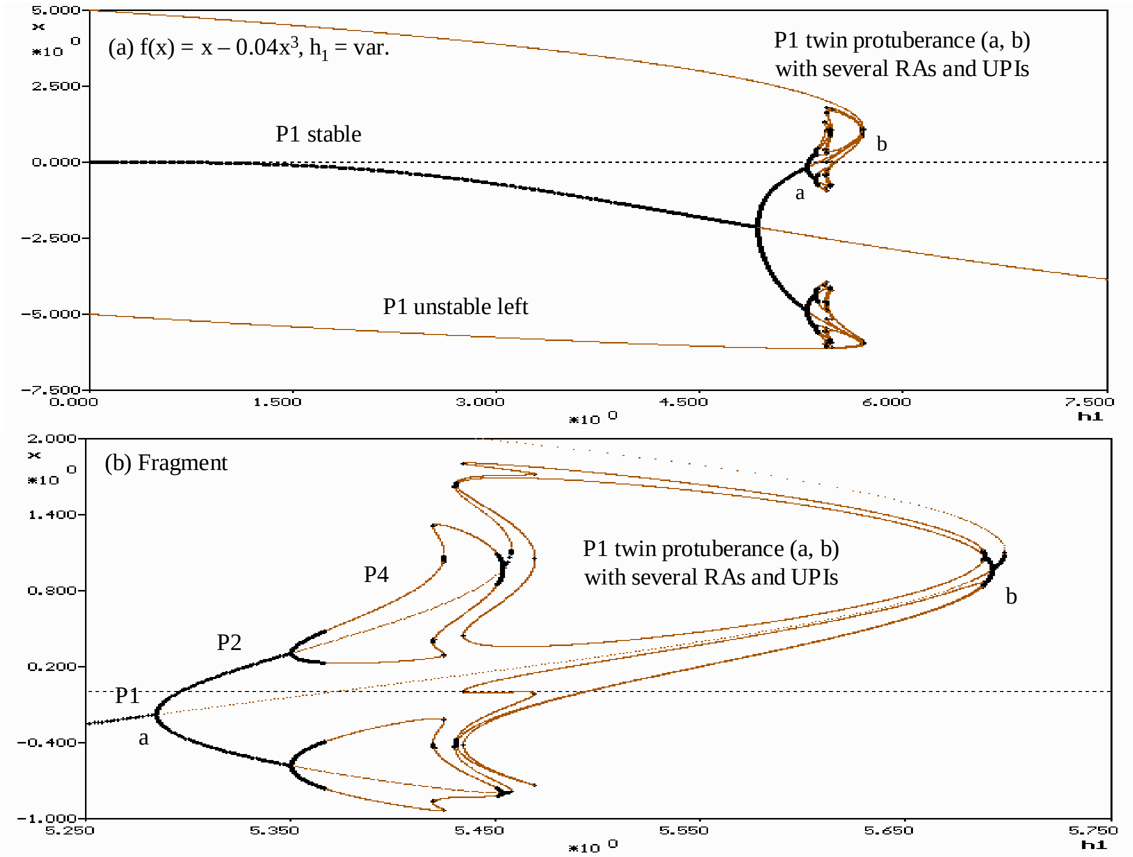


Fig. 18 Duffing symmetrical soft harmonically driven system with large dissipation. 1T bifurcation group with P1, P2, and P4 stable (dark) and unstable (reddish) solutions. The system has very complex twin protuberances with rare attractors (RAs) and unstable periodic infinitiums (UPIs). Parameters: $f(x) = x + a_3x^3$, $a_3 = -0.04$, h_1 = var., w = 1, b = 1.0.

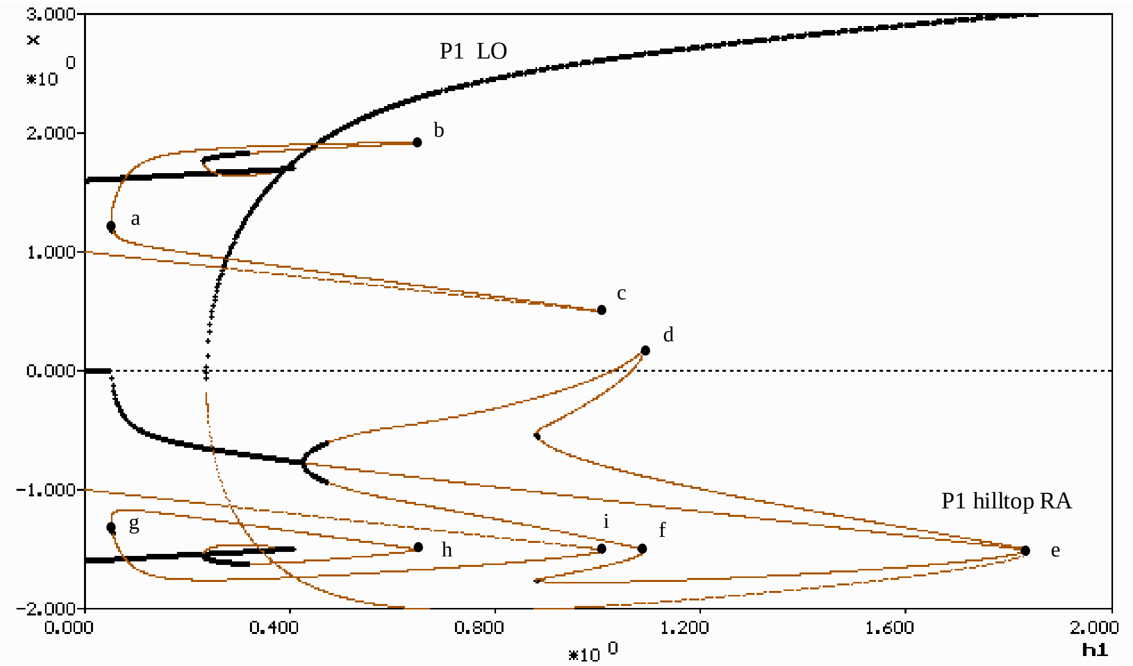


Fig. 19 Five-linear symmetrical driven system with three potential wells. Bifurcation diagram for three 1T bifurcation groups. P1 symmetrical and P1 asymmetrical twin stable and unstable solutions are shown in the figure. The system has many rare attractors (RA) marked by black circles: a, b,…, g, h. Inside the each marked circle there are rare bifurcation sub-groups with period-doubling cascade and rare chaotic attractors. Parameters: c1 = 1, c2 = - 1, c3 = 5, d1 = 0.5, d2 = 1.5, b = 0.1, w = 1, h1= var.

8. The role of asymmetry in nonlinear driven systems

Introducing the asymmetry in symmetrical systems leads to additional complexity of bifurcation diagrams. It may be reason of birth of chaotic parameter regions and complex protuberances (see Figs. 10,14).

9. Homoclinics and chaotic attractors, and chaotic transients

It is possible to show that homoclinics in the driven damped nonlinear systems appears if the system has subharmonic isles with unstable periodic infinitiums. This conclusion do more concrete the common theorems received by Poincaré and Birkhoff (see Figs. 7-9).

10. Systems with several equilibrium positions. Pendulum systems

The method of complete bifurcation groups allows to find new unexpected phenomena in driven damped pendulum systems and in the systems with several equilibrium positions. Earlier it was found that it is possible to find stable periodic solutions near unstable equilibrium positions in the driven systems [39,42-44,49]. The results received by the method of complete bifurcation groups show that there are stable hilltop solutions not only period-1, but more complicate subharmonic attractors, for example, period-3 or period-4 and even chaotic hilltop attractors. These conclusions are right for different systems such as Duffing soft system, systems with two, three and so on potential wells, and for different driven and parametric pendulum systems (Figs. 16-19). Such systems have many rare attractors and parameter regions with UPIs and chaos.

11. The theory of rare dynamical phenomena and rare attractors

We suppose that the concepts of rare attractors and the method of complete bifurcation groups may be put in the foundation of the theory of rare dynamical phenomena. Some important applications of the theory based on the method of complete bifurcation groups expect to our mind in celestial mechanics: dynamics of natural and artificial satellites, in the three-body and n-body problems [38,40]; medicine and biology; dynamics of new nanomaterials; nonlinear economics and others.

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References

[1] **Poincaré H.** Mémoire sur les cóurbes definies par les équations différentielles I-VI, Oeuvre I, Gauthier-Villar, Paris, 1880-1890.

[2] **Poincaré H.** Sur les équations de la dynamique et le problème des trois corps, Acta Math., 13, 1-270, 1890.

[3] **Poincaré H.** Les Méthodes Nouvelles de la Mécanique Celeste, 3 Vols, Gauthier-Villars, Paris, 1899.

[4] **Birkhoff G.D.** Dynamical Systems, A.M.S. Publications, Providence, 1927.

[5] **Andronov A.A., Leontovich E.A., Gordon I.I., and Maier A.G.** Theory of Bifurcations of Dynamic Systems on a Plane, Israel Program of Scientific Translations, Jerusalem, 1971.

[6] **Andronov A.A., Leontovich E.A., Gordon I.I., and Maier A.G.** Theory of Dynamic Systems on a Plane, Israel Program of Scientific Translations, Jerusalem, 1973.

[7] **Andronov A.A., Pontryagin L.** Systèmes Grossiers, Dokl. Akad. Nauk. SSSR, 14, 247-251, 1937.

[8] **Andronov A.A., Vitt E.A., and Khaiken S.E.** Theory of Oscillators, Pergamon Press, Oxford, 1966.

[9] **Guckenheimer J. and Holmes P.** Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields, Springer-Verlag, New York, Applied Mathematical Sciences, Volume 42, 1983.

[10] **Shaw S.W., Holmes P.J.** A periodically forced piecewise linear oscillator, J. of Sound and Vibration, 90(1):129-155, 1983.

[11] **Blekhman I.I.** Vibrational Mechanics, World Scientific, Singapore, (Russian original: Nauka, Moscow, 1994), 2000.

[12] **Thompson, J.M.T. and Stewart, H.B.** Nonlinear Dynamics and Chaos, 2nd Edition, Wiley, 2002.

[13] **Ueda Y.** The Road to Chaos - II, Aerial Press Inc. - Santa Cruz, 2001.

[14] **Landa P.S.,** Regular and Chaotic Oscillations, Springer-Verlag Berlin, 2001.

[15] **Malinetskii G.G., Potapov A.B.** Nonlinear Dynamics and Chaos, Moscow, 2006 (in Russian).

[16] **Kuznetsov A.P., Kuznetsov S.P., Ryskin N.M.** Nonlinear Oscillations, Fizmatlit, Moscow, 2002 (in Russian).

[17] **Magnitskii N.A., Sidorov S.V.** New Methods of Chaotic Dynamics, Moscow, 2004 (in Russian).

[18] **Thomsen J.J.** Vibrations and Stability, 2nd edition, Springer-Verlag, Berlin, 2003.

[19] **Leonov G.A.** Strange Attractors and Classical Stability Theory, St.Petersburg University Press, 2008.

[20] **Moon F.C.** Chaotic Vibrations, Wiley, 2004.

[21] **Trubetskov D.I.** Introduction to synergetics. Chaos and Structure, Moscow, 2004 (in Russian).

[22] **Kuznetsov Yu.A.** Elements of Applied Bifurcation Theory (third edition), Springer-Verlag, New York, Applied Mathematical Sciences, Volume 112, 2004.

[23] **Feigin M.I.** Forced Oscillations in Systems with Discontinuous Nonleariries, Nauka, Moscow, 1994 (in Russian).

[24] **Leine R.I. and van Campen D.H.** Discontinuous fold bifurcations in mechanical systems, Archive of Applied Mechanics, 72 (2-3):138-146, April 2002.

[25] **Mosekilde E. and Zhusubalyev Zh.** Bifurcations and Chaos in Piecewise-smooth Dynamical Systems, World Scientific, 2003.

[26] **Wiercigroch M., Sin V.W.T., Li K.** Measurement of chaotic vibration in symmetrically piecewise linear oscillator, J. of Chaos, Solitons and Fractals, 9(1-2):209-220, 1998.

[27] **di Bernardo M., Budd C.J., Champneys A. R., Kowalczyk P.** Piecewise-smooth Dynamical Systems. Theory and Applications, Springer-Verlag, London, Applied Mathematical Sciences, Volume 163, 2008.

[28] **Luo A.C.J., Menon S.** Global chaos in a periodically forced, linear system with a dead-zone restoring force, J. of Chaos, Solitons and Fractals 19(5):1189-1199, 2004.

[29] **Naess A., Mo E.** A numerical study of the existence and stability of some chaotic attractors by path integration, in this issue of J. of Vibroengineering.

[30] **Bishop S.R.** Impact oscillators, Phil. Trans. Roy. Soc. A, 347:347-351, 1994.

[31] **Nordmark A.B.** Non-periodic motion caused by grazing incidence in impact oscillators, J. of Sound and Vibrations, 2:279-297, 1991.

[32] **Astashev V.K., Babitsky V.I.** Ultrasonic cutting as a nonlinear (vibro-impact) process, Ultrasonics, Nr.6, pp. 89-96, 1998.

[33] **Peterka F., Vacik J.** Transition to chaotic motion in mechanical systems with impacts, J. of Sound and Vibration, 154(1):95-115, 1992.

[34] **Hogan S.J.** On the dynamics of rigid-block motion under harmonic forcing, Proc. Roy. Soc. London A, 425:441-476, 1989.

[35] **Hogan S.J.** Damping in rigid block dynamics contained between side walls, J. of Chaos, Solitons and Fractals, 11:495-506, 2000.

[36] **Lakshmanan M., Rajasekar S.** Nonlinear Dynamics. Integrability, Chaos, and Patterns, Springer-Verlag, Berlin, 2003.

[37] **Ragulskis K., Ragulskis L.** Dynamical directivity and chaos in some systems with supplementary degrees of freedom, in this issue of J. of Vibroengineering.

[38] **Cartmell M.P., D'Arrigo M.C.** Simultaneous forced and parametric excitation of a space tether, In proceedings of the 33rd Summer School on Advanced Problems in Mechanics, Russian Academy of Sciences, St.Petersburg, pp. 347-355, 2005.

[39] **Zevin A.A.** Some amazing phenomena in stability of nonlinear dynamical systems, Proceedings of the 6th European Nonlinear Dynamics Conference ENOC-2008, 4 p., 2008.

[40] **Beletsky V.V., Rodnikov A.V.** On evolution of libration points similar to eulerian in the model problem of the binary-asteroids dynamics, in this issue of J. of Vibro-engineering.

[41] **Zakrzhevsky M.** The Nonlinear Way of Thinking and Education Problems in Nonlinear Dynamics, Proceedings of 2nd European Nonlinear Oscillations Conference, Prague, Vol.2, 1996, pp. 255-260.

[42] **Zakrzhevsky M., Frolov V.** Bifurcation Analysis of the Stable Forced Oscillations near Unstable Equilibrium Position, Proceeding of the 2nd European Nonlinear Oscillations Conference, Prague, 1996, Vol.2, pp. 265-267.

[43] **Zakrzhevsky M.** Global Stable Oscillations Near Unstable Equilibrium Positions: The Hilltop Effect, in F.C. Moon (ed.), Proceedings of the IUTAM Symposium on New Applications of Nonlinear and Chaotic Dynamics in Mechanics, held in Ithaca, USA, Kluwer Academic Publishers, 1998, pp. 117-124.

[44] **Zakrzhevsky M.** Stable Forced Hilltop Oscillations, in E. Lavendelis and M. Zakrzhevsky (eds.), Proceedings of the IUTAM Symposium on Synthesis of Nonlinear Dynamical Systems, held in Riga, Latvia, 1997, Kluwer Academic Publishers, 2000, pp. 285-294.

[45] **Zakrzhevsky M.** Nonlinear Oscillatory and Vibro-Impact Systems: Rare Attractors, in V.K.Astashev and V.L.Krupenin (eds.) The Dynamics of Vibroimpact (Strongly Nonlinear) Systems, Russian Academy of Sciences, Moscow-Zvenigorod, 2001, pp. 156-162.

[46] **Schukin I.T., Zakrzhevsky M.V., Ivanov Yu.M., Kugelevich V.V., Malgin V.E., Frolov V.Yu.** Realization of Direct Methods and Algorithms for Global Analysis of Nonlinear Dynamical Systems, in this issue of J. of Vibro-engineering.

[47] **Yevstignejev V.Yu., Zakrzhevsky M.V., Schukin I.T.** Bifurcation analysis by method of complete bifurcation groups of the driven system with two degrees of freedom with three equilibrium positions, in this issue of J. of Vibro-engineering.

[48] **Smirnova R.S., Zakrzhevsky M.V., Yevstignejev V.Yu., Schukin I.T.** Paradoxes of increasing linear damping in the nonlinear driven oscillators, in this issue of J. of Vibro-engineering.

[49] **Frolov V.** Improvement of Method and Algorithms for Simulation of Strongly Nonlinear Oscillators, Ph.D Thesis, 1997.

[50] **Smirnova R.S.** Regular and Chaotic Forced Oscillations in Piecewise Linear Systems with Nonlinear Dissipation, Ph.D Thesis, RTU, Riga-Daugavpils, 2002.

[51] **Schukin I.T.** Development of the methods and algorithms of simulation of nonlinear dynamics problems. Bifurcations, chaos and rare attractors, PhD Thesis, Riga – Daugavpils, 2005.

[52] **Yevstignejev V.Yu.** Application of the Complete Bifurcation Groups Method for Analysis of Strongly Nonlinear Oscillators and Vibro-Impact Systems, PhD thesis, Riga, 2008.