390. SLOW STEADY-STATE FLOW OF VISCOUS FLUID NEAR VIBRATING CYLINDER; ANALYTICAL SOLUTION FOR ASYMMETRIC CHANNEL

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Abstract. High frequency oscillations of a circular cylinder in a viscous fluid near a fixed plane are investigated. Solution of an ideal fluid uniform flow, when velocity at the infinity is given, is presented. This solution is supplemented by a circulary flow and the circulatory constant is calculated approximately. Potential fluid flow velocity on the circle surface is displayed in diagrams and in trigonometric series.

Key words: Ultrasound, oscillations, viscous fluid, ideal fluid, circulatory flow.

1. Introduction

When a cylindrical body executes high frequency swing oscillations in a viscous fluid, flow of this fluid can be resolved into two principle parts. The first is oscillations of the fluid at the same frequency as the body, the second has no periodic component and is steady-state creeping flow. This flow can be deduced by applying boundary layer solution, presented by Schlichting [1]. The Schlichting's solution can be carried out if flow of an ideal fluid round the cylinder is determined. Analytical solution of the ideal plane fluid flow in a channel with a circular cylinder, placed symmetrically in the middle of the channel, is presented in [2]. Investigations of such flows can be applied in medicine, surgical technologies, ultrasonic angioplasty.

In this paper solution of the same flow is expressed for a circular cylinder, placed in the vicinity of a boundary of the channel. Circulation round the cylinder is investigated and approximate evaluation of this flow is suggested. This flow disapears naturally when the cylinder is in the middle of the channel.

2. Flow through channel in potential field

General solution of a potential fluid flow in a channel with circular cylinde, placed asymmetricaly, can be expressed by mapping conformally the doubly connected fluid flow domain to circular ring of an auxiliary complex variable ς.

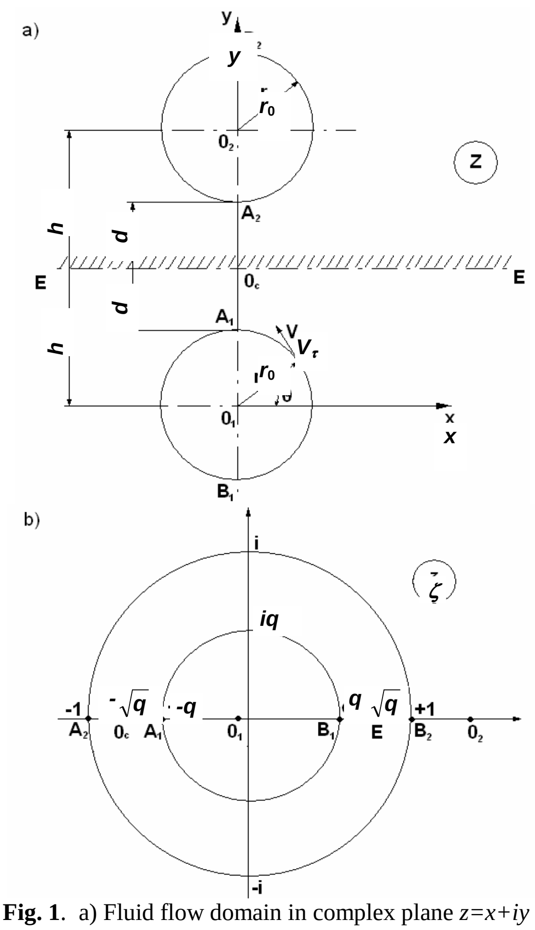

Fig. 1. a) Fluid flow domain in complex plane $z=x+iy$; b) parametric ζ plane

Applying elliptical functions the boundary value problem can be reduced to a singly connected problem, but this solution is complicated. If the cylinder is located closely to one of the channel boundaries, influence of the remote boundary can be neglected. Therefore the fluid flow domain can be presented as infinite plane $z=x+iy$ with two equal circles (Fig. 1a). Uniform flow in this plane has a relatively simple solution, recently presented by Crowdy [3]. Conformal mapping of a fluid flow domain to the unit circle in the parametric ς plane (Fig. 1b) is realized by linear fractional function

$$z = ai\frac{\zeta+\sqrt{q}}{\zeta-\sqrt{q}} + hi \ ,$$

where constants a, q depend on radius r_0 and distances between the discs and the centers:

$$a = \sqrt{d\left(h+r_o\right)} \ , \quad q = \frac{h-a}{h+a} \ .$$

From the inverse function

$$\zeta = \sqrt{q}\,\frac{z+(a-h)i}{z-(a+h)i} \tag{1}$$

one can see that entire imaginary axis y of the z plane is mapped to the real axis of the ς plane. The middle point of the flow O_c and the infinite point E are mapped to $-\sqrt{q}$ and $+\sqrt{q}$, while images of the centers O_1 and O_2 of the discs in the z plane are $\zeta=-q\sqrt{q}$ and $\zeta = 1/\sqrt{q}$. As always $q<1$, the images O_1 and O_2 are inside the circle $|\zeta|=1$. The function

$$P(\zeta,q) = (1-\zeta)\prod_{k=1}^{\infty}\left(1-q^{2k}\zeta\right)\left(1-q^{2k}\zeta^{-1}\right)$$

and logarithmic derivative

$$\zeta\frac{P_\zeta}{P} = \frac{\zeta}{\zeta-1} + \sum_{k=1}^{\infty}\left(\frac{q^{2k}\zeta}{q^{2k}\zeta-1} + \frac{q^{2k}}{\zeta-q^{2k}}\right) \equiv K(\zeta,q)$$

are expressed. As $\beta=\sqrt{q}$ [3], two functions $K_1 = K\left(\zeta/\sqrt{q},q\right)$, $K_2 = K\left(\sqrt{q}\zeta,q\right)$ can be defined. The complex potential of the uniform flow, when infinite point speed U parallel to the y axis, is

$$W_1(\zeta,\beta) = \frac{U}{\sqrt{q}}\left(K_1-K_2\right) \ . \tag{2}$$

When infinite point speed U parallel to the x axis the complex potential is

$$W_2(\zeta,\beta) = \frac{U}{i\sqrt{q}}\left(K_1+K_2\right) \ . \tag{3}$$

The complex velocity

$$\frac{dW_2}{dz} = \frac{d\zeta}{dz}\frac{U}{i\sqrt{q}}\left(\frac{dK_1}{d\zeta}+\frac{dK_2}{d\zeta}\right) \ , \tag{4}$$

where

$$-\frac{dK_1}{d\zeta} = \frac{\sqrt{q}}{\left(\zeta-\sqrt{q}\right)^2} + \sum_{k=1}^{\infty}\frac{\sqrt{q}}{q^{2k}}\left[\frac{1}{\left(\zeta-\sqrt{q}q^{-2k}\right)^2} + \frac{q^{-2k-1}}{\left(q^{-2k}\zeta/\sqrt{q}-1\right)^2}\right] ,$$

$$-\frac{dK_2}{d\zeta} = \frac{1}{\sqrt{q}\left(\zeta-1/\sqrt{q}\right)^2} + \sum_{k=1}^{\infty}\frac{1}{\sqrt{q}q^{2k}}\left[\frac{1}{\left(\zeta-q^{-2k}/\sqrt{q}\right)^2} + \frac{q^{-2k-1}}{\left(\sqrt{q}q^{-2k}\zeta-1\right)^2}\right] ,$$

The derivative $d\zeta/dz$ can be deduced from Eq.(1).

The line $y=h$ in Fig. 1a is a simmetry line and a streamline also, so it can be replaced by the channel border. Streamlines for uniform potential flow in a half-plane past a circular cylinder are in Fig. 2 (Eq. (3)). The streamlines past the same two cylinders , when flow U is perpendicular to the flow in Fig. 2 , is presented in [3] and can be deduced from Eq. (2).

3. Circulatory flow

Solution of the flow past a cylinder, located in vicinity of a infinite wall, presented in the chapter 2, is not unique. Any circulatory flow can be added to this uniform flow because velocity of this flow vanishes in infinity. Motion of a vortex near some obstacles is investigated by Jonson, McDonald [4]. Solution of a circulatory flow at the wall is depicted by Milne-Thomson in Chapter 6, [5]. The complex potential of the flow in the domain below the line $y=h$ in Fig. 1a is

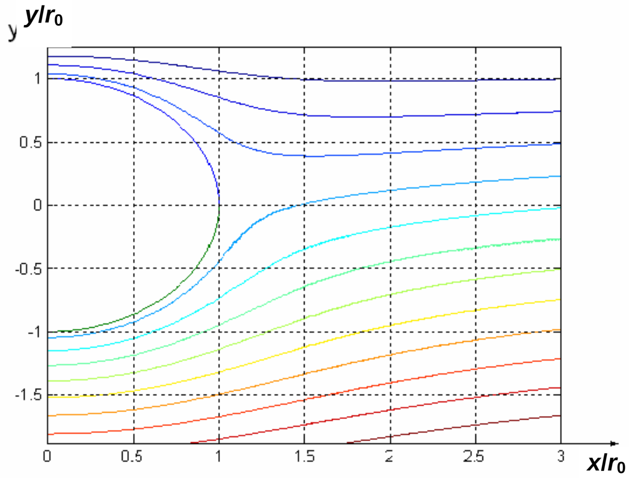

Fig. 2. Streamlines of uniform potential flow past a cylinder when d=0.25 r_o

$$w = \kappa \, arc\cot\frac{z - hi}{c}\,, \tag{5}$$

where c and κ - some constants. Applying $w = \varphi + i\psi$, where velocity potential $\varphi = \varphi(x, y)$ and stream function $\psi = \psi(x, y)$ are real functions, Eq. (5) can be rearanged to

$$\frac{u_c + iv_c}{1 - iu_c v_c} = \frac{c}{x + iy - ih}\,. \tag{6}$$

The variables $u_c = \tan\frac{\varphi}{\kappa}, v_c = \tanh\frac{\psi}{\kappa}$ are real also. From two real equations, deduced from (6), the function u_c can be eliminated:

$x^2 + (y - h)^2 + c(v_c + v_c^{-1})(y - h) + c^2 = 0$. As $v_c = const$ when $\psi = const$, this is the streamline equation – a circle of a radius $r_o = c(v_c - v_c^{-1})/2$, when center coordinates of the circle are $y_c = h - c(v_c + v_c^{-1})/2$, $x_c = 0$. It can be deduced that the constant $c = \sqrt{h^2 - r_o^2}$. Streamlines for the circulatory flow are in Fig. 3.

Complex velocity of the circulatory flow

$$\frac{dw}{dz} = \frac{\kappa c}{(z - hi)^2 + c^2}\,, \tag{7}$$

and velocity of the uniform flow (4) in the gap O_cA_1 (Fig. 1a) are compared with the circulatory flow velocity in Fig. 4.

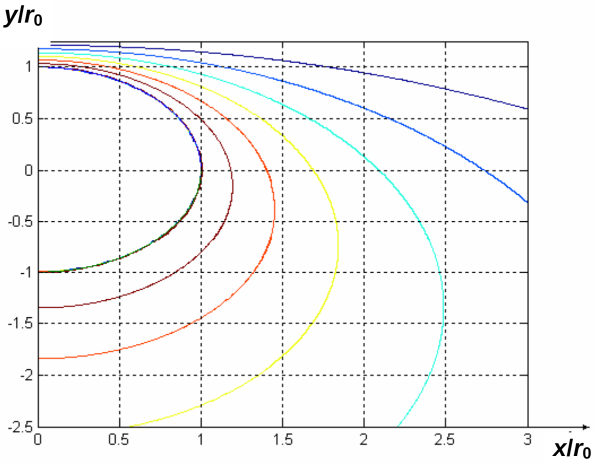

Fig. 3. Streamlines of circulatory flow when d=0.25 r_o

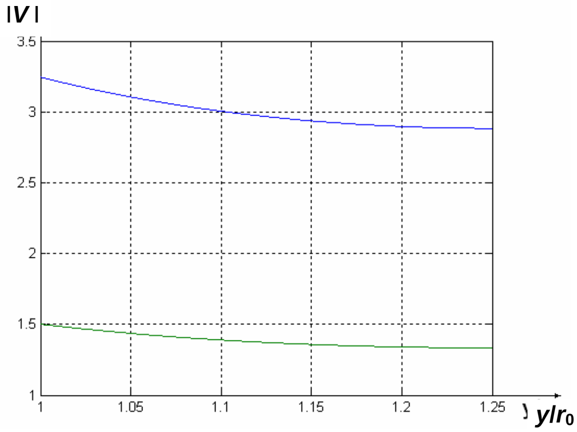

Fig. 4. Velocity $v = |v_x|$ on the y axis between the circular cylinder and the channel border when d=0.25 r_o, κ = 1 and velocity of the uniform flow $v = v_x = -1$ when $x \to \infty$

Velocities $v = |v_x|$ on the channel border and $|v_x|$ on the x axis are depicted in Fig. 5.

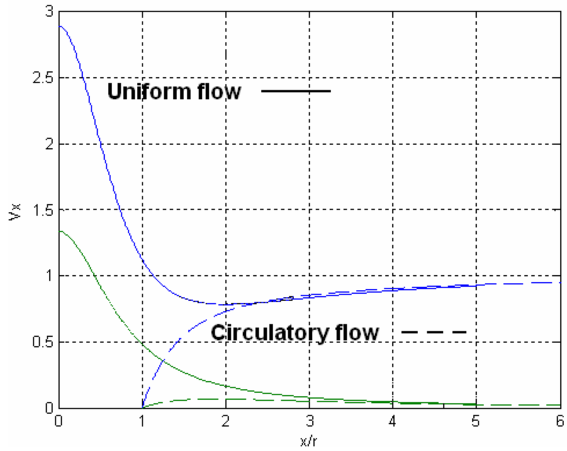

Fig. 5. Velocity $|v_x|$ when d=0.25 r_o, κ = 1 on the channel border y=h (continuous lines) and on the x axis (dashed lines)

The circulatory flow velocity decreases when $x \to \infty$ and this can be easily seen from Eq. (7). When cyrcular cylinder is in the middle of the channel [2] the circulatory flow vanishes naturally. But when fluid flow domain is asymmetric the circulatory flow can be generated. This is widely known in the wing flow theory of ideal flow [5] and can be determined for the viscous fluid [6].

If the gap between the cylinder and the channel border is narrow, influence of a fluid boundary layer can be significant. Longitudinal oscillations of a plane generates displacements of the fluid [1]

$$u(y,t) = u_o e^{-ky} \cos(\omega t - ky) \ , \ k = \sqrt{\frac{\omega}{2\nu}} \ , \qquad (8)$$

where y is axis, perpendicular to the plane, ω - frequency, ν - kinematic viscosity. Thickness of the layer $\delta \approx \sqrt{\frac{\nu}{\omega}} = 0.96 \times 10^{-3} cm$, where dynamic viscosity of blood $\mu = 23 \times 10^{-3}$ Pa×s, frequency f = 40 kHz ($\nu = 0.23$ cm^2 /s , $\omega = 8 \pi \times 10^4$ rad/s).

Amplitude of the velocity in the Eq. (8) $|\dot{u}| = b\omega e^{-ky}$. The area under the curve $A = \int_0^l |\dot{u}| dy = \frac{b\omega}{k}\left(1 - e^{-kl}\right)$, therefore mean value of the velocity in the interval [0,l] $|\dot{u}|_m = \frac{b\omega}{kl}\left(1 - e^{-kl}\right)$.

Ratio of the alteration of the velocity to the infinity velocity $|\dot{u}|_\infty = b\omega$ is

$$S_a = \frac{1 - e^{-kl}}{kl} \qquad (9)$$

and depends only on dimensionless parameter $kl = \frac{l}{\sqrt{2}\delta}$. When distance $l=d$ (i.e. the gap, Fig. 1a) is very narrow $kl \to 0$ then $S_a \to 1$. When the gap is wide $kl \to \infty$ then $S_a \to 0$. The velocity of the uniform and the circular flows in the gap vary little(Fig.4). The correction parameter κ in Eq. (5) can be approximated as $\kappa_a = S_a \frac{v_{um}}{v_{cm}(1)}$, where v_{um} is mean square velocity of the uniform flow, $v_{cm}(1)$ - mean square velocity of the circulary flow when κ =1.

If velocity $\dot{u}(y,t)$ as function of distance and time from Eq. (8) is applied, the area $A = A(t)$ and then mean square area with respect to time $A_m = \sqrt{\frac{1}{T}\int_0^T A^2(t)dt}$, $T = \frac{2\pi}{\omega}$ determined, the ratio

$$S_m = \frac{\sqrt{\left(1 - e^{-kl}\cos kl\right)^2 + e^{-2kl}\sin^2 kl}}{2kl} \qquad (10)$$

can be deduced. The mean square ratio $S_m < S_a$ for any value of kl, but the difference is the most significant when $kl \to 0$ as $S_m \to \sqrt{2}/2$, while $S_c \to 1$.

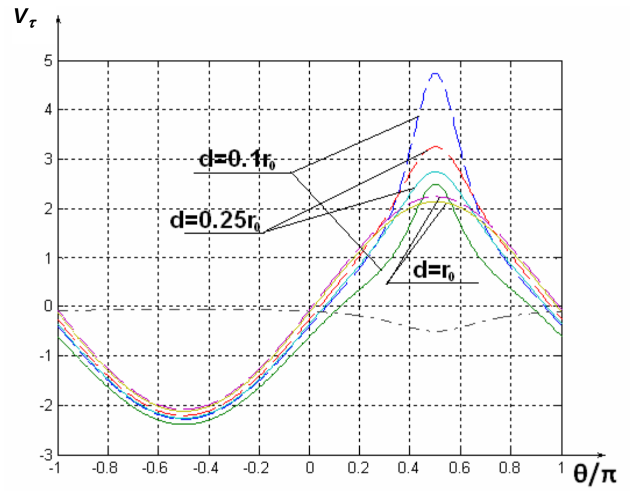

Fig. 6. Dependence of fluid velocity tangent component on angle θ for uniform flow without circulation (dashed lines) , uniform circulatory flow (continuous lines) , circulatory flow when d=0.25 r_o , κ = 1 (dashed-dotted line)

In Fig. 6 tangent component of the fluid flow velocity v_τ on the circular cylinder as a function of angle θ (Fig. 1a) are depicted. The dasded lines are for the uniform flow when circulation is absent, the continues lines when uniform flow and circulatory flow are summed. The circulatory flow is adjusted by $\kappa_m = S_m v_{um}/v_{cm}(1)$. One can see from the diagrame that influence of the circulation is more important when the gap d is less. What's more, if the gap $d \to 0$ the velocity of the uniform flow with circulation in the gap ($\theta \approx \pi/2$) is decreasing while velocity of the uniform flow without circulation is increasing. Over the rest of the circle surface ($\theta < 0.2\pi$, $\theta > 0.8\pi$) difference between the curves is approximately constant and when $\theta = -\pi/2$,velocity of the fluid is increasing, when the gap is decreasing, for all flows. Velocity of the circulatory flow has the same direction over the whole circle and its modulus increases in the vicinity of the gap.

Position θ_o of the zero velocity point $v_\tau = 0$ depends on the gap d . The angle θ_o increases when the distance d decreases for all flows (the uniform flows without circulation and the uniform circulatory flows also). If the gap d is a constant, the angle θ_o is larger for a uniform circulatory flow. Obviously $\theta_o = 0$ when the circle is in the middle of a channel [2].

It is advantageous to present the fluid velocity v_τ as trigonometric polinomial

$$v_\tau = b_o + \sum_{k=1}^{n} \left(a_k \sin k\theta + b_k \cos k\theta \right) . \qquad (11)$$

Table 1. Factors of the trigonometric polynomial, Eq. (11)

d/r_0	0.1	0.25	1.0
b_0	-0.489	-0.168	-0.058
a_1	2.008	2.249	2.104
b_1	-0.235	-0.264	-0.059
a_2	-0.231	-0.178	
b_2	0.174	0.104	
a_3	0.119	0.056	
b_3	-0.078		
a_4	-0.050		
b_4	0.032		

The factors a_k , b_k depend on the ratio d/r_o (Table 1). The series (11) in the table is truncated Fourier series when relative mean square error is less than 2%.

In all solutions the constant U was given a value that the infinite velocity $v_x \to 1$ when $x \to \infty$.

4. Dicussion

When a velocity at the infinity point is given a potential plane uniform flow past a cylinder has no unique solution. The additional arbitrary parameter can be the velocity circulation around the cylinder. Velocity of the circulatory flow at the infinity is zero (Eq. (7), Fig. 5). There is no reason why this circulation flow should be absent generally, and viscosity of the fluid have to be taken into consideration to assess the circulation.

When the gap d between the circle and the border of the channel diminishes (Fig. 1a), velocity of the potential fluid in the gap increases if the circulation is absent. This dependence is really likely when the gap d is much more than the thickness δ of the boundary layer. Influence of the viscosity will alter this if $d \approx \delta$. A radius r_o of the circle could be increased to $\approx r_o + \delta$, but velocity in the narrower gap is even greater. Obviously adding a circulatory flow is more realistic way to assess the viscosity of the fluid, and still retain in potential field. The two circulation approximations Eq. (9) and Eq. (10), suggested in this paper, possibly can be improved considering two parallel planes at a distance d [7]. On the other hand, there is only one plane in reality and d is the minimal gap distance.

Position of the zero velocity point θ_o on a circular cylinder strongly depends on the gap distance d and a circulation flow. The other zero velocity point, where streamlines divide, is $\theta_o^* = \pi - \theta_o$, symmetric with respect to the y axis.

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