374. ON PARAMETRIC OSCILLATIONS OF PENDULAR AND ROTOR SYSTEMS

A. M. Gouskov and G. Ya. Panovko
Mechanical Engineering Research Institute of RAS
4, Malii Kharitonievsky per., Moscow, 101990, Russia
E-mail: *gam@ostrov.net*, *gpanovko@yandex.ru*

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Abstract. Different kind of parametrical phenomena in vibrating systems are presented. Nonlinear properties of rotating and pendulum dynamical systems give some interesting industrial applications. The purpose of this paper is to show the possibility to generalize the models and to find a similarity of certain physical presentations and possible analysis. The effects of non-uniform rotation of unbalanced rotor, synchronization of multiples pendulums with common vibrating base, stabilization of vertical column, auto parametrical damper etc. are included in consideration.

Keywords: pendular and rotor elements, dynamic stabilization, parametric systems stability, selfsynchronization, autoparametric vibration suppression

Nomenclature: Effects of non linear mechanics in vibration systems

Introduction

Among the huge variety of vibrational machinery containing pendular and rotor elements and of possible description approaches for studies thereof, let us consider only those related with parametric oscillations [1-3]. The parametric excitation of oscillations in pendular and rotor systems results in a number of effects that are widely used to solve various vibrational or manufacturing problems [4].

Many of these effects are closely related. For example, upraise of irregularity in rotation of a rotor mounted on a vibrating foundation, self-synchronization of two or more pendula, vibrational sustaining of rotation of an unbalanced rotor appear in presence of the dynamic linkages of the same type.

The appearance of dynamic stability of the upper position of a pendulum, dynamic stabilization of the vertical axis of a flexible rod curved under the effect of gravity, stabilization of the position of a free body mounted on a vertical rod are also related with one another and can be explained based on the theory of the parametric systems stability.

The described models and phenomena can be associated with the pendular systems models featuring the autoparametric resonances. In this study we consider the above mentioned effects on the example of a model to which many real structures can be reduced.

Studied system/Computational model

An absolutely rigid body m_0 (main or carrying body) is fixed on a linear viscoelastic suspension with stiffness c and viscous damping d (**Fig. 1**). The ideal guide allows the body to realize only translational motion along the vertical axis Oy which originates at the static equilibrium position of the center of gravity of the carrying body.

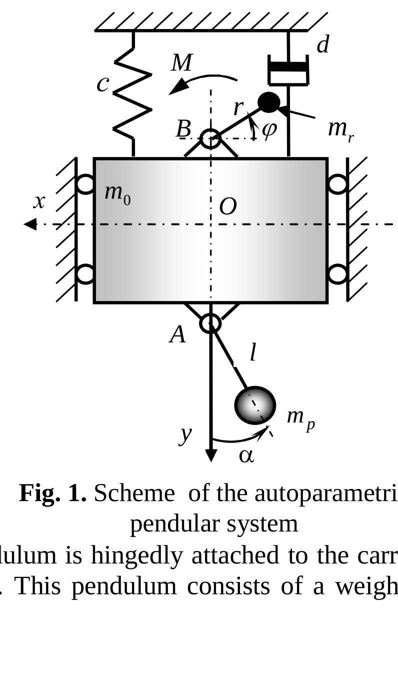

Fig. 1. Scheme of the autoparametric pendular system

A pendulum is hingedly attached to the carrying body in point A. This pendulum consists of a weightless rigid bar of length *l*, and of a lump mass m_p situated on the free end of it. The rotations axis of the pendulum is normal to the plane *xOy* of the working element motion. The pendulum can effect angular oscillations, characterized by an angular degree of freedom α measured from the static equilibrium position of the pendulum. A deviation of the pendulum engenders a restoring couple of linear resistance $S = d_1 \dot{\alpha}$, with d_1 coefficient of resistance to rotation.

An unbalance vibration exciter (unbalanced rotor) with mass m_r and eccentricity r is installed on the main body in point *B*. The spin axis of the unbalanced rotors, as well as the axis of the pendulum, is perpendicular to the plane *xOy* of the carrying body motion. The plane *xOy* can be vertical or horizontal (in the latter case, gravity terms are excluded from the equations of motion).

The unbalance position relative to the working element will be defined as the angular co-ordinate φ of t radius vector r of the gravity center of the unbalance measured from the vertical axis *Oy*. The rotation of the unbalance is sustained by an external torque *M (*electric engine torque), which can be described by the steady-state characteristic of the engine. For steady motions the external torque can be assumed constant. In general, we will consider the motion of the system under the effect of gravity, although in some special cases gravity will be neglected.

The system (**Fig. 1**) consists of three partial subsystems: main body on the viscoelastic suspension with motionless pendulum and non-rotating exciter $\{y \neq 0, \varphi = \alpha \equiv 0\}$, the unbalance of the exciter with the motionless main body $\{\varphi \neq 0,\ \alpha = 0,\ y = 0\}$ and the pendulum with the motionless main body. $\{y \equiv 0, \varphi = 0,\ \alpha \neq 0\}$.

The equations of the motion are following:

$$\begin{cases} m\,\ddot{y} + d\dot{y} + c\,y - m_r r\left(\ddot{\varphi}\sin\varphi + \dot{\varphi}^2\cos\varphi\right) - \\ - m_p l\left(\ddot{\alpha}\sin\alpha + \dot{\alpha}^2\cos\alpha\right) = 0; \\ J_p\ddot{\alpha} + d_1\dot{\alpha} - m_p l(\ddot{y} - g)\sin\alpha = 0; \\ J_r\ddot{\varphi} - m_r r(\ddot{y} - g)\sin\varphi = M; \end{cases} \tag{7}$$

with $m = m_0 + m_p + m_r$ - total mass of the whole system., where y is displacement of the main body from the static equilibrium along the downwards direction *Oy*, α, φ - unbalance and pendulum angular co-ordinates.

The system (7) describes the non-linear oscillations of the autoparametric system with three degrees of freedom. Let us consider the particular cases following from the analysis of this system.

The Rotor rotation irregularity, vibrational sustaining of rotation, self-synchronization

Irregularity of the unbalance rotation caused by the rotation axis oscillations. For the sake of simplicity, assume the frequencies of the system are far removed from parametric resonances of the pendulum. In this case the pendulum does not oscillate $(\alpha \equiv 0)$. Assume also that the main body constraints are ideal (i.e. $c = 0$ and $d = 0$) and the gravity induces no torque.

Therefore, in the steady state, the external moment is null and from the equations (7) it follows:

$$\begin{cases} m\,\ddot{y} = m_r r(\ddot{\varphi}\sin\varphi + \dot{\varphi}^2\cos\varphi); \\ J_r\ddot{\varphi} - m_r r\,\ddot{y}\sin\varphi = 0. \end{cases} \tag{8}$$

By eliminating from (8) the terms containing the main body acceleration $\ddot{y}$, we obtain the second order nonlinear equation relative to the sought unbalance angle of rotation function:

$$\ddot{\varphi} - \lambda^2(\ddot{\varphi}\sin\varphi + \varphi^2\cos\varphi) = 0,$$

$$\text{with} \quad \lambda = m_r r\big/\sqrt{J_r m} < 1. \tag{9}$$

For $t = 0: \varphi = 0$ and $\dot{\varphi} = \dot{\varphi}_0,\ y = -y_0$, and $\dot{y} = 0$ we obtain from (9):

$$\dot{\varphi} = \dot{\varphi}_0\Big/\sqrt{1 - \lambda^2\sin^2\varphi}\ . \tag{10}$$

Because of the periodicity, the angular velocity (10) can be represented by the Fourier series:

$$\dot{\varphi} = \dot{\varphi}_0(a_0 - a_2\cos 2\varphi + a_4\cos 4\varphi - ...), \tag{11}$$

with $a_0,\ a_2, ...,\ a_n$ expansion coefficients of the angular velocity $\dot{\varphi}$ [2].

In steady state the constant component in (11) is to be equal to the average angular velocity ω of the unbalance:

$$\dot{\varphi}_0 a_0 = \omega, \quad \text{therefore} \quad \dot{\varphi}_0 = \omega/a_0\,. \tag{12}$$

Then the series (11) may be rewritten in the following form:

$$\dot{\varphi} = \omega - \tilde{a}_2\cos 2\omega t + \tilde{a}_4\cos 4\omega t - ...,, \tag{13}$$

$$\text{with} \quad \tilde{a}_2 = \omega a_2/a_0,\ \tilde{a}_4 = \omega a_4/a_0\ .$$

Therefore, under oscillatory motion of the axis of rotation, the angular velocity of the unbalance is a periodic function only composed by even harmonic terms.

The oscillations of the angular velocity of the unbalance influence, naturally, the oscillations of the carrying body. From the first equation of (8) it follows

$$\ddot{y} = m_r r\left(\ddot{\varphi}\sin\varphi + \dot{\varphi}^2\cos\varphi\right)\big/m\,. \tag{14}$$

Since $\sin\varphi = \sin\omega t$ and substituting the series (13) in (14) we obtain the expression of the carrying body acceleration:

$$\ddot{y} = m_r r\omega\left(q_1\cos\omega t + q_3\cos 3\omega t + q_5\cos 5\omega t + ...\right)\big/m\,. \tag{15}$$

where coefficients $q_1,\ q_3,\ q_5, ...$ are combinations of the coefficients $\tilde{a}_2,\ \tilde{a}_4, ..$ [2].

Therefore, the oscillations of the carrying body contain not only the fundamental frequency but also an infinite set of odd harmonics.

It would be interesting to notice that the irregularity of rotation of the unbalance may also be caused by the variation of the moment of gravity forces if the unbalance rotates about a horizontal axis. For the sake of clarity, assume that the carrying body is motionless and its displacement is $y \equiv 0$. This case corresponds to the free rotation of the unbalance about a fixed axis. As before, we consider the steady state when the moment in the undamped system is null. The second equation of the system (14) gives then:

$$J_r \ddot{\varphi} + m_r rg \sin\varphi = 0; \tag{16}$$

or

$$\ddot{\alpha} + \Omega^2 \sin\alpha = 0, \tag{17}$$

with $\Omega = \sqrt{m_r rg / J_r}$ - natural frequency of small oscillations of the unbalance around its static equilibrium position.

By integrating (17) with initial conditions corresponding to the lower position of the unbalance gravity center $t = 0,\ \varphi = 0,\ \dot{\varphi} = \dot{\varphi}_0$, we obtain:

$$\dot{\varphi} = \sqrt{\dot{\varphi}_0^{\,2} - 2\Omega^2 (1 - \cos\varphi)}. \tag{18}$$

Whence it follows that the angular velocity is not constant, but varies between $\dot{\varphi}_{\max} = \dot{\varphi}_0$ at $\varphi = 2\pi n$ and $\dot{\varphi}_{\min} = \dot{\varphi}_0 \sqrt{1 - \lambda^2}$ at $\varphi = (2n+1)\pi$; with n=0,1,2, … ; $\lambda = 2\Omega / \dot{\varphi}_0 < 1$.

Usually the magnitudes of the harmonic components of the angular velocity of the unbalance caused by its axis oscillations appears to be greater then the oscillations due to the gravity forces moment variation.

Therefore, the unbalance angular velocity oscillates about its mean position which is constant (or varying slowly). The rotation irregularity results in the additional harmonic components in the spectrum of the excitation force.

Usually these oscillations of the angular velocity are relatively small, but in some cases they can have a pronounced effect on system dynamics. On the other hand, provided the appropriate choice of system parameters, these oscillations can be amplified and used for some practical objectives, e.g. in order to generate a super harmonic drive of a vibrational machine [2].

Vibrational sustaining of rotation. Assume that under the effect of given external moment M the unbalance of the vibration exciter spins with a constant rotation speed $\omega = \omega_0 = const$. Then the rotation angle is $\varphi = \omega t$ and its derivatives are $\dot{\varphi} = \omega,\ \ddot{\varphi} = 0$. As a result, the first and the second equations of (7) may be rewritten in the following form:

$$\begin{cases} m\ddot{y} + d\dot{y} + c\,y = m_r r\omega^2 \cos\omega t + \\ + m_p l\left(\ddot{\alpha}\sin\alpha + \dot{\alpha}^2 \cos\alpha\right); \\ J_p \ddot{\alpha} + d_1 \dot{\alpha} - m_p l(\ddot{y} - g)\sin\alpha = 0; \end{cases} \tag{19}$$

and the third one will determine the value of the drive torque, necessary to sustain the carrying body and pendulum oscillations, i.e. $-m_r r(\ddot{y} - g)\sin\omega t = M$.

The right-hand term of the first equation of (19) includes a nonlinear component due to the influence of the pendulum oscillations on the main body motion. In order to simplify the analysis, this influence can be neglected as well as the dynamical reactions of the stays of the main body. Thus, pendulum axis motion can be considered as periodic: $y = A\cos\omega t$.

Then the second equation of (19) becomes the following:

$$J_p \ddot{\alpha} + d_1 \dot{\alpha} + m_p l(g - A\omega^2 \cos\omega t)\sin\alpha = 0. \tag{20}$$

In the simplest case the pendulum rotates with a steady angular velocity ω, i.e.

$$\alpha = \omega t + \gamma(t), \tag{21}$$

where $\gamma(t)$ stands for an additional periodic component of the rotation angle, the average value of which over a period is null.

The function (21) is not an exact solution of (20) because the latter will not be satisfied at arbitrary t. However, if one takes an average of the equation (20) over a period $2\pi/\omega$ with (21) taken into account, it comes out that the pendulum rotation is possible under the condition that the periodic component of its rotation angle can be described by the following expressions [4,5]:

$$\sin\gamma = 2d_1 / \left(m_p lA\omega\right) \quad \text{or} \quad 2d_1 / \left(m_p lA\omega\right) < 1. \tag{22}$$

Self-synchronization. The phenomenon of self-synchronization consists in the existence of rotation of the pendulum and the unbalance fixed on the main body (**Fig. 1**), with equal angular velocities while their partial angular velocities are different [4,6]. In other words the case is to find the conditions of existence and stability of solutions of the system (7) as following:

$$\begin{aligned} &y = y(\omega t),\ \alpha = \sigma_1\left(\omega t + \varepsilon_1 + \gamma_1(\omega t)\right), \\ &\varphi = \sigma_2\left(\omega t + \varepsilon_2 + \gamma_2(\omega t)\right). \end{aligned} \tag{23}$$

Here ω is the absolute value of the synchronous angular velocity of the pendulum and the unbalance, ε_i are initial rotation phases of the pendulum and the unbalance (constant), $\sigma_i = \pm 1$ characterizes the direction of rotation, $y(\omega t),\ \gamma_i(\omega t)$ are 2π-periodic functions of ωt.

The laws of motion (23) suppose that the pendulum and the unbalance rotation have a phase lag with respect to the carrying body oscillations. As in the case of vibrational rotation sustaining, the solution (23) would not satisfy (7) for every t. However, when the process is considered as an average over the period $2\pi/\omega$, i.e. one rotation of the pendulum or of the unbalance, the solutions (23) can be considered as approximate, which provide the expressions of the synchronous rotation speed ω and of the phase lag $\varepsilon = \varepsilon_1 - \varepsilon_2$ [2,4,5].

Dynamic stability under parametric excitation (dynamic stability of the inverse pendulum)

From the equation (20) under small angular deviations of the pendulum (**Fig. 2,a**) from the vertical (for $\sin\alpha \approx \alpha$) it follows [1]:

$$J_p\ddot{\alpha} + d_1\dot{\alpha} + m_p l(g - A\omega^2 \cos\omega t)\alpha = 0 .$$

After adimensionalizing the equations we come down to Mathieu's equation. The stability analysis shows that the upper position of the pendulum is attained under the condition $\omega > \sqrt{2gl}/A$. Notice that damping brings about the appearance of a threshold of the excitation beyond which the stability is achieved.

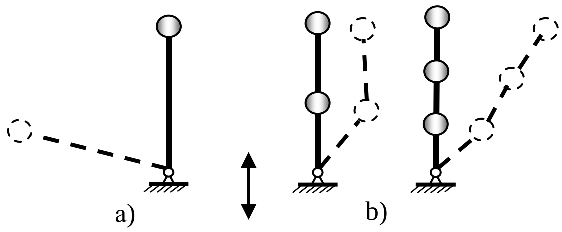

Fig. 2. Kpitsa's pendulum (a) multipendular system (b)

P.L. Kapitsa was one of the first to observe this phenomenon experimentally and to provide its theoretical interpretation [7]. In numerous works that followed other researchers have shown that vertical vibrations can stabilize chain pendula (**Fig. 2, b**) [7].

Further insight into the dynamical stability of the chain pendula allowed to explain the effect of appearance of stability of vertical axis of flexible rods and strings under vibration. We provided a detailed presentation of this field while the last conference [8]. There are various examples of flexible structures with a close-to-zero transverse stiffness (antennae, hoses, ropes) attain a straight vertical axis under vibrations.

Autoparametric vibration suppression

All the above mentioned examples (except the dynamical stabilization of a flexible rod) correspond to non-resonant vibration excitation. It is of particular interest to consider the case of resonance excitation of main body vibrations while the natural frequency of the pendulum oscillations is tuned to the frequency of the principal parametric resonance (i.e. in the ratio 1:2 to the main resonance). Then the model given by Fig. 1 can be taken as an autoparametric system with a dynamic vibration dampener realized by the pendulum [9,10]. This system dynamics are also described by the equations (7).

In order to simplify the analysis, the variance of the unbalance rotation velocity will be neglected: $\varphi = \omega t$ with $\omega = const$, which allow us not to consider the interaction of the main body with the exciter. Therefore, it can be inferred from (7):

$$\begin{cases} m\ddot{y} + d\dot{y} + c\,y = \\ = ml\left(\ddot{\alpha}\sin\alpha + \dot{\alpha}^2\cos\alpha\right) + m_r r\omega^2 \sin\omega t; \\ J_p\ddot{\alpha} + d_1\dot{\alpha} + m_p gl\sin\alpha = m_p l\ddot{y}\sin\alpha. \end{cases} \quad (24)$$

For further computations we will use scales and adimensional parameters, issued from partial subsystems consideration.

The time scale T_* and linear displacement scale Y_* can be taken respectively as the period of the natural oscillations of the main body on viscoelastic suspension while the pendulum is motionless and static deviation y_0 of the main body with the pendulum mass, i.e.:

$$T_* = \sqrt{m/c},\ Y_* = mg/c . \quad (26)$$

Then the adimensional time and displacement of the main body are written as follows

$$\tau = t/T_*,\ \xi = y/Y_* . \quad (27)$$

The normalization coefficient of linear damping for the degree of freedom ς can be expressed [11] as

$$\varsigma = d/\left(2\sqrt{cm}\right) < 1 \quad (28)$$

(ς =1 corresponds to critical damping).

After introducing an adimensional pendulum length

$$\lambda = l/Y_* , \quad (30)$$

one can express the normalized natural frequency of the pendulum small oscillations ($\sin\alpha \approx \alpha$):

$$\beta = T_*\sqrt{g/l} \quad \text{or view of (3) and (7):} \quad \beta = 1/\sqrt{\lambda} . \quad (31)$$

The normalization coefficient of linear damping for the degree of freedom α ca be written as:

$$\varsigma_1 = d_1/\left(2m_p l\sqrt{gl}\right) < 1 \quad (32)$$

($\varsigma_1 = 1$ corresponds to critical damping).

In order to adimensionalize the equations (24), the following dimensionless groups will also be introduced:

$$\mu = m_p/m,\ \Omega = \omega T_*,\ Q = m_r r/\left(mY_*\right), \quad (33)$$

The independent analysis of the influence of the exciter mass m_r, radius r and of the pendulum mass m_p, another normalization carried out via another normalization and other dimensionless groups, e.g.:

$$\varsigma_0 = d/\left(2\sqrt{cm}\right),\ \mu_p = m_p/m,\ \mu_r = r/Y_*,\ \rho = r/Y_* . \quad (34)$$

The ratio $2\beta = 1$ corresponds to the internal resonance in the autoparametric system $\omega_\alpha : \omega_\xi = 1:2$. Moreover, if the frequency of the external forcing equals the natural frequency of the natural system $\Omega = 1$, the external resonance takes place.

Results

The results of numerical computations of non-dimensional magnitudes of oscillations of the main body A_ξ^0 and the pendulum angle B_α^0 as functions of the excitation frequency (FRF) are given on **Fig. 3** and **Fig. 4** by solid lines. As a comparison, the FRF with the pendulum switched off is given by the dashed line on **Fig. 3**. All the computations are conducted with the following values of the parameters:

$\beta = 0{,}5; \mu_r = 0{,}04; \mu_p = 0{,}15; \rho = 0{,}5;\ \zeta_0 = 0{,}02;\ \zeta_1 = 0{,}01.$

The tuning of the pendulum on the internal resonance $\beta = 0{,}5$ in the vicinity of the external resonance $\Omega \approx 1$ provokes a substantial reduction of magnitude (more than ten fold) of oscillation of the main body (solid line) as compared to the resonant response, traced for the motionless pendulum (dashed line). It is important to notice that the pendulum damper is efficient only in the neighborhood of the resonance peak.

Depending on whether the frequency increases or decreases, different segments of the response curve take place. As the frequency Ω increases, the segments ABCDEF are realized, and the segments FGHJA as it decreases (on **Fig. 3** and **Fig.4** arrows show the magnitude on the response curve branches that are realized under respective frequency Ω variation trend). As may be inferred from **Fig. 4**, the pendulum triggering occurs in a jump-like manner.

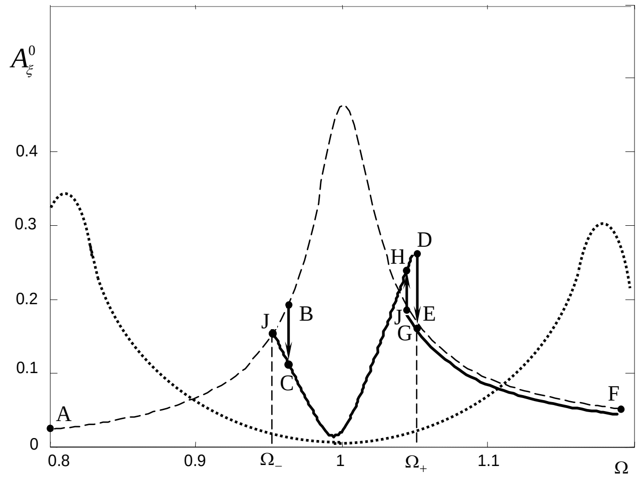

Fig. 3. Principal masse magnitude characteristic

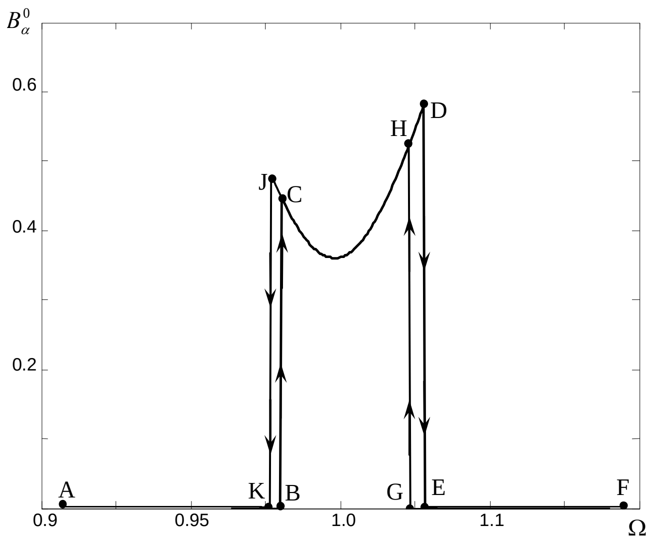

Fig. 4. Pendulum magnitude characteristic

The discontinuous behavior of the resonance curve near the resonance implies drastic conditions of turning on and off of the pendulum dampener and is associated with a torque of resistance to the pendulum rotation. Outside the interval $\Omega \in [\Omega_-, \Omega_+]$ the pendulum remains motionless (**Fig. 4**) and will not bring about any additional resonances.

Conclusions

The revealed properties of autoparametric vibration dampener show that its application is more efficient in comparison with a linear dampener which is a tuned mass damper [11]. The use of the pendulum dampener is particularly efficient in systems subject to broad-band excitations, when it is important to reduce the magnitude of the specific resonant oscillations. The autoparametric damper operates only in the vicinity of the tuned frequency and does not imply any resonant oscillations on other frequencies, in contrast to classical dampeners, the efficient performance is accompanied by occurrence of resonances on other frequencies (dashed line on **Fig. 3**). Notice that other conditions being equal (aside from structural particularities and strength considerations) the classical damper might be somewhat more efficient. However, the presence of additional resonance peaks makes this type of dampeners less suitable, especially in case of brad-band excitation.

Acknowledgments

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