364. Nonlinear effects related to vibrations of long elastic waveguides: formulation of nonlinear equations

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Abstract. A number of effects arising during vibrations of long elastic waveguides can not be explained in the context of the linear theory. One of these effects is redistribution of energy between the longitudinal and transverse vibration modes. Nonlinear equations describing such effects have been studied in momentless approximation with application to vibrations of strings and they account for tension of the string axis as a result of transverse vibrations. The authors have studied a possibility of generalization of these equations for the case of elastic waveguides taking into account internal moments and lateral forces arising during bending of the waveguide.

Keywords: flexible waveguide, ultrasound, longitudinal vibration, flexural vibration

Introduction

Today flexible elastic waveguides for transmission of ultrasonic vibrations are found increasingly wide application in different areas of science and technology, e.g. ultrasonic thrombolysis [1, 2], transurethral lithotripsy [3], heating of fuel at low temperatures [4], remote actuation of ultrasonic motors [5], cleaning of difficult-to-access channels in technical systems. Unfortunately at present there are no methods for designing such waveguides and their analysis and synthesis are implemented by empirical way. Vibrations of flexible waveguides are of complex nature and should be treated as coupled flexural-longitudinal vibrations.

Combined longitudinal and flexural vibrations of ultrasonic systems have been considered previously in the works by Zhou *et al.* [6, 7]. However longitudinal and flexural vibrations were considered as independent on each other.

An attempt for mathematical modeling of flexible waveguides has been made in the work by Bansevečius *et al.* [8] in which flexural vibrations of a waveguide with a constant cross-sectional area along the length have been considered. Unfortunately the results presented in this work cannot be generalized for the case of waveguides with a complex law of the cross-sectional area variation along the length.

The problem of the flexible waveguides modeling is also considered in the article by Gavin *et al.* [9] in which finite-element model of a waveguide immersed into fluid is studied. Although this model makes it possible to study waveguides with arbitrary complex law of the cross-sectional area variation along the length, it is based on some assumptions reducing its practical value. Particularly the problem is considered to be axisymmetric so only longitudinal vibrations can be studied. At the same time appearance of flexural vibrations essentially reduces efficiency of ultrasound transmission along the large-length waveguides and therefore should be taken into consideration during design.

In practice longitudinal and flexural vibrations of long flexible waveguide cannot be treated independently. If initially there are only longitudinal vibrations, as it takes place in the case of attachment of a waveguide to a solid horn, they can be transformed into flexural vibrations if the waveguide is sufficiently long and looses its dynamic stability. It means that a portion of longitudinal vibrations energy is transferred to flexural mode of vibration and constant-energy solutions obtained from the independent treatment of equations of longitudinal and flexural modes cannot adequately describe vibration of the waveguide. Flexural vibrations at the distal end of the waveguide should be minimized and it may be achieved by means of attachment of pear-shaped working ending to the distal end. In this case flexural vibrations originating from the loss of dynamic stability are transformed into longitudinal vibrations and it means that longitudinal and flexural modes are exchanged with energy. As the loss of dynamic stability is a typically nonlinear phenomenon, coupling of longitudinal and flexural vibrations should be described by nonlinear equations. Nonlinear equations of coupled longitudinal-flexural vibration are well developed for strings [10, 11]. However they are based on assumption that the only kind of internal forces arising in the string are tensional stresses (momentless theory). Flexural vibrations affect longitudinal ones by means of creating additional tensional stresses in the string caused by deformation of the string axis. Consideration of this deformation leads to geometric nonlinearity.

Nonlinear equations of elastic bar movement are studied by Hsieh *et al.* [12] and can be useful for deriving equations of flexible waveguide vibration. These equations account for the deformation of the bar axis and include geometric nonlinearity. However elastic bar studied by Hsieh *et al.* has constant cross-sectional area and equations of its movement should be generalized to adequately describe vibration of flexible waveguide.

Formulation of nonlinear equations of flexible waveguide vibration

Let us consider a segment of the waveguide having coordinate x and length dx in undeformed state. Displacement of this segment during deformation is shown in the Fig. 1. Cross section of the waveguide is rotated by the angle $\alpha = \theta + \gamma$, where θ is bending angle, γ is shear angle.

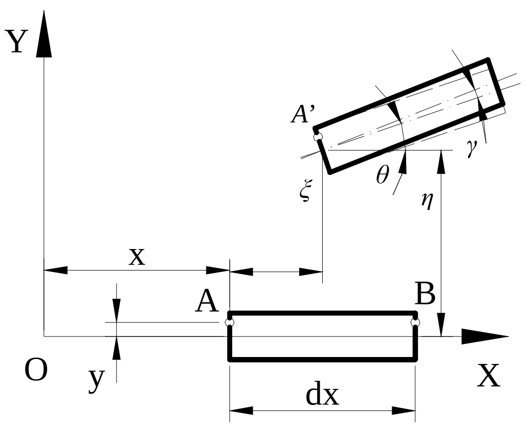

Fig. 1. Displacement of the waveguide segment during deformation

Every point $A(x, y)$ of the waveguide cross section is translated during deformation into the point $A'(\bar{x}, \bar{y})$ with coordinates

$$\bar{x} = x + \xi - y\sin\theta ,$$

$$\bar{y} = \eta + y\cos\theta ,$$

where ξ and η are longitudinal and transverse displacements respectively.

Length of the element $A(x, y)B(x+dx, y)$ after deformation can be defined from equation

$$ds = \sqrt{d\bar{x}^2 + d\bar{y}^2} =$$

$$= \sqrt{(1 + \xi' - y(\sin\theta)')^2 + (\eta' + y(\cos\theta)')^2}\, dx ,$$

where accent stands for the partial derivate with respect to the variable x.

Relative deformation of the element in the linearized form is given by equation

$$\varepsilon = \frac{ds}{dx} - 1 = \sqrt{\eta'^2 + (1+\xi')^2} - 1 - y\theta' = \varepsilon_0 - y\theta' ,$$

where ε_0 is relative deformation of the waveguide axis.

Material of the waveguide is assumed to be elastic and the following relations are satisfied:

$$\sigma_{xx} = E\varepsilon , \ \sigma_{xy} = G\gamma ,$$

where E is modulus of elasticity, G is shear modulus.

Axial force N, bending moment M and shear force Q are expressed by equations

$$N = \int_S \sigma_{xx} dS = ES\varepsilon_0 , \qquad (1a)$$

$$M = \int_S \sigma_{xx} y \cdot dS = -E\theta' \int_S y^2 dS = -EI\theta' , \qquad (1b)$$

$$Q = \int_S \sigma_{xy} dS = GS\gamma , \qquad (1c)$$

where I is centroidal moment of inertia of the waveguide section, S is cross-sectional area of the waveguide.

Equations of the segment motion can be derived from the equilibrium conditions:

$$(N\cos\alpha)' - (Q\sin\theta)' = \rho S\ddot{\xi} , \qquad (2a)$$

$$(N\sin\alpha)' + (Q\cos\theta)' = \rho S\ddot{\eta} , \qquad (2b)$$

$$M' - Q(1+\varepsilon_0)\cos\gamma = -J\ddot{\theta} , \qquad (2c)$$

where J is mass moment of inertia of the waveguide per unit length.

From the Eq. (2c)

$$Q = \frac{M' + J\ddot{\theta}}{(1+\varepsilon_0)\cos(\alpha - \theta)} = \frac{(M' + J\ddot{\theta})(1+\varepsilon_0)^{-1}}{\cos\alpha\cos\theta + \sin\alpha\sin\theta} .$$

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Angle α is related to the displacements ξ and η by equations

$$\sin\alpha = \frac{\eta'}{1+\varepsilon_0}, \; \cos\alpha = \frac{1+\xi'}{1+\varepsilon_0}. \qquad (3)$$

Taking into account Eqs. (3) and Eq. (1b) gives the following relation

$$Q = \frac{J\ddot{\theta} - EI'\theta' - EI\theta''}{(1+\xi')\cos\theta + \eta'\sin\theta}. \qquad (4)$$

After insertion of Eqs. (3) and (4) Eqs. (2a) and (2b) take the following form

$$E(S'(1+\xi') + S\xi'')\left(1 - \frac{1}{\sqrt{\eta'^2 + (1+\xi')^2}}\right) + ES(1+\xi')\times$$

$$\times\frac{\eta'\eta'' + (1+\xi')\xi''}{(\eta'^2 + (1+\xi')^2)^{3/2}} - \left(\frac{J\ddot{\theta} - EI'\theta' - EI\theta''}{(1+\xi')\cot\theta + \eta'}\right)' = \rho S\ddot{\xi}, \qquad (5a)$$

$$E(S'\eta' + S\eta'')\left(1 - \frac{1}{\sqrt{\eta'^2 + (1+\xi')^2}}\right) + ES\eta'\times$$

$$\times\frac{\eta'\eta'' + (1+\xi')\xi''}{(\eta'^2 + (1+\xi')^2)^{3/2}} + \left(\frac{J\ddot{\theta} - EI'\theta' - EI\theta''}{(1+\xi') + \eta'\tan\theta}\right)' = \rho S\ddot{\eta}. \qquad (5b)$$

From Eqs. (3)

$$\tan\alpha = \frac{\eta'}{1+\xi'}.$$

On the other hand

$$\tan\alpha = \tan(\theta+\gamma) = \tan\left(\theta + \frac{Q}{GS}\right).$$

Taking into account Eq. (4) we obtain the following relation

$$\tan\left(\theta + \frac{J\ddot{\theta} - EI'\theta' - EI\theta''}{GS((1+\xi')\cos\theta + \eta'\sin\theta)}\right) = \frac{\eta'}{1+\xi'}. \qquad (5c)$$

Eqs. (5a)-(5c) form the basis for mathematical description of the waveguide vibration. These equations can be solved by means of asymptotic perturbation methods, e.g. method of multiple scales [12, 13].

Conclusions

In this article we have derived equations of flexible waveguide vibration taking into account coupling of longitudinal and flexural modes. To our knowledge this is the first attempt to account for nonlinear modal interaction in complex-mode ultrasonic vibratory systems. The future work in this direction will be aimed at numerical analysis of the suggested equations and development of theoretical basis for creation of flexible waveguide systems with controlled spatial distribution of longitudinal and flexural vibrations.

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