311. Representation of Time Averaged Vibrating Images in the Operator Format

Z. Navickas, M. Ragulskis
Faculty of Fundamental Sciences, Studentų 50, Kaunas University of Technology, Kaunas LT-51368, Lithuania
E-mail: *zenonas.navickas@ktu.lt, minvydas.ragulskis@ktu.lt*

(Received 10 September 2007; accepted 18 October 2007)

Abstract. The process of time averaging of greyscale images is expressed in the operator format. It is shown that the inverse problem of the reconstruction of the original image from its time averaged image is an ill-posed problem. Application of time averaging techniques is proposed for cryptographic applications especially when the density function of variable defining the dynamic deflection from the state of equilibrium is arcsine distribution.
Keywords: time averaging, vibrating images, operator format.

1. Introduction

Construction of time averaged images of objects when those objects or camera (or both) perform some type of motions is a classical research area with numerous applications in science and engineering. A typical application is the removal of blur caused by undesirable motions of object or camera [1].

Inverse problems involving blur removal are characterized by the fact that the information of interest (the distribution of greyscale colour intensity on the surface of an analyzed body) is not directly available. The imaging device (the camera) provides measurements of a transformation of this information in the process of time averaging. In practice, these measurements are both incomplete (sampling) and inaccurate (statistical noise). This means that one must give up recovering the exact image. Indeed, aiming at full recovery of the information usually results in unstable solutions due to the fact that the reconstructed image is very sensitive to inevitable measurement errors. In other words, slightly different data would produce a significantly different image. In order to cope with these difficulties, the reconstructed image is usually defined as the solution of an optimization problem [2].

Completely different are experimental optical time average techniques. The object of these techniques is not removal of blur caused by camera shake, but interpretation of digital images produced by oscillation of structures [3, 4]. Particularly, such techniques find important applications when moiré fringe patterns are considered [5].

The object of this paper is the mathematical principles of the formation of time averaged images. We develop the necessary mathematical formulations and express the formation of digital averaged images in the operator format. That helps to interpret the inverse problems and to justify the applicability of arcsine distribution for cryptographic applications.

2. Definitions of Time Averaging

Definition 1. Function $F(x)$ is a greyscale function if it is defined for all $x \in R$ and satisfies the following conditions:

a) $0 \le F(x) \le 1$;

b) has only a finite number of discontinuity points;

c) $\int_{-\infty}^{+\infty} \left| F(x) - \frac{1}{2} \right| dx < +\infty$;

d) piecemeal continuous in any finite interval $[-l, l]$; $l > 0$;

e) limit $\lim_{x \to \pm\infty} F(x) = \frac{1}{2}$ exists.

If 0 corresponds to black colour and 1 corresponds to white colour, the "background" colour at infinity is grey.

Definition 2. Time averaging operator H_s for harmonic oscillations is defined as [6]:

$$H_s F(x) = \frac{1}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} F(x + s \sin t) dt, \tag{1}$$

where s is the amplitude of harmonic oscillations; $s \geq 0$; $x \in R$.

Definition 3. Arcsine density function of a random variable ζ_s is:

$$p_s(x) = (\mathbf{1}(x+s) - \mathbf{1}(s-x)) \frac{1}{\pi\sqrt{s^2 - x^2}}, \quad s > 0, \qquad (2)$$

where $\mathbf{1}(v) = \begin{cases} 1, \text{when } v \geq 0; \\ 0, \text{when } v < 0; \end{cases}$ is unitary Heaviside function.

Explicitly,

$$p_s(x) = \begin{cases} \dfrac{1}{\pi\sqrt{s^2 - x^2}}, \text{when} -s < x < s; \\ 0 - \text{elsewhere.} \end{cases} \qquad (3)$$

Lemma 1. If the density function of a random variable ζ_s is arcsine density, then $E\zeta_s^0 = 1$; $E\zeta_s^{2k-1} = 0$;

$$E\zeta_s^{2k} = \frac{(2k-1)!!}{(2k)!!} s^{2k}, \ k = 1, 2, \ldots.$$

Proof

The first equality is trivial. The second one follows from the fact that distribution function of ζ_s is symmetric with respect to y-axis. The third equality can be proved changing the variables $x = s \sin t$:

$$E\zeta_s^{2k} = \frac{1}{\pi} \int_{-s}^{s} \frac{x^{2k}}{\sqrt{s^2 - x^2}} dx =$$

$$= \frac{1}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{s^{2k} (\sin t)^{2k}}{s \cos t} s \cos t dt = \qquad (4)$$

$$= \frac{2}{\pi} s^{2k} \frac{(2k-1)!!}{(2k)!!} \frac{\pi}{2} = \frac{(2k-1)!!}{(2k)!!} s^{2k},$$

where $(2j-1)!! = 1 \cdot 3 \cdot \ldots \cdot (2j-1)$; $(2j)!! = 2 \cdot 4 \cdot \ldots \cdot (2j)$; $j = 1, 2, \ldots$.

End of Proof.

Let Φ denote Fourier transform, and Φ^{-1} - inverse Fourier transform. Then:

$$\Phi f(x) = \int_{-\infty}^{+\infty} \exp(-ivx) f(x) dx = \hat{f}(v);$$

$$\Phi^{-1} \hat{f}(v) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \exp(ivx) \hat{f}(v) dv = f(x). \qquad (5)$$

It can be noted that when $f(x)$ is a symmetric real function, its Fourier transform is a real function.

Lemma 2. If $p_s(x)$ is arcsine distribution, its Fourier transformation is $\hat{p}_s(v) = J_0(sv)$, where $J_0(z)$ is zero order Bessel function of the first kind.

Proof

$$\hat{p}_s(v) = \int_{-\infty}^{+\infty} p_s(x) e^{-ixv} dx =$$

$$= \sum_{k=0}^{+\infty} \int_{-\infty}^{+\infty} \frac{(-ixv)^k}{k!} p_s(x) dx =$$

$$= \frac{2}{\pi} \sum_{k=0}^{+\infty} \frac{(-1)^k v^{2k}}{(2k)!} \frac{(2k-1)!!}{(2k)!!} \frac{\pi}{2} s^{2k} =$$

$$= \sum_{k=0}^{+\infty} (-1)^k \frac{(vs)^{2k}}{(2k!!)^2} = \sum_{k=0}^{+\infty} (-1)^k \left(\frac{1}{k!} \left(\frac{vs}{2} \right)^k \right)^2 =$$

$$= J_0(sv),$$

where $(2k)!! = 2^k k!$; $(2k-1)!!(2k)!! = k!$.

End of proof.

Thus, the characteristic function of the arcsine distribution is zero order Bessel function of the first kind (the characteristic function is Fourier transformation of the density function of the random variable defining the deflection from the state of equilibrium).

3. Main Theorems

Theorem 1. $H_s F(x) = EF(x + \zeta_s)$. (6)

Proof

Change of variables $s \sin t = y$ is exploited in the proof.

Then, $dt = \dfrac{dy}{\sqrt{s^2 - y^2}}$, and:

$$\frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} F(x+s\sin t)dt = \frac{1}{\pi}\int_{-s}^{s} F(x+y)\frac{dy}{\sqrt{s^2-y^2}} = \qquad (7)$$
$$= EF(x+\zeta_s).$$

End of proof.

Corollary 1. Equality $F(x+\zeta_s)=F(x-\zeta_s)$ holds true because the arcsine density function $p_s(x)$ is a symmetric function and therefore satisfies equality $p_s(x)=p_s(-x)$.

Theorem 2. Let $F(x)=\sum_{k=0}^{+\infty} a_k \frac{x^k}{k!}$; for all x and given $a_k \in R$. Then,

$$H_s F(x) = \sum_{j=0}^{+\infty}\left(D_x^{2j}F(x)\right)\left(\frac{1}{j!}\left(\frac{s}{2}\right)^j\right)^2, \qquad (8)$$

where D_x^n is the operator of differentiation (x is the variable of differentiation, n is the order of differentiation); $s \geq 0$.

Proof

The following identities are exploited in this proof:

a) $\sum_{l=0}^{+\infty} a_{l+2j}\frac{x^l}{l!} = D_x^{2j}\left(\sum_{l=0}^{+\infty} a_l \frac{x^l}{l!}\right)$,

b) $\frac{1}{(-n)!} = 0$ for $n = 1,2,\ldots$.

$$\frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} F(x+\sin t)dt = \frac{1}{\pi}\sum_{k=0}^{+\infty}\frac{a_k}{k!}\int_{-s}^{s}\frac{(x+y)^k}{\sqrt{s^2-y^2}}dy =$$
$$= \sum_{k=0}^{+\infty}\frac{a_k}{k!}\sum_{j=0}^{k}\binom{k}{j}x^{k-j}\frac{1}{\pi}\int_{-s}^{s}\frac{y^j}{\sqrt{s^2-y^2}}dy =$$
$$= \sum_{k=0}^{+\infty}\sum_{j=0}^{+\infty}\binom{k}{2j}\frac{a_k}{k!}x^{k-2j}\frac{(2j-1)!!}{(2j)!!}s^{2j} =$$
$$= \sum_{j=0}^{+\infty}\sum_{k=0}^{+\infty}a_k\frac{k!}{(2j)!(k-2j)!}\frac{1}{k!}\frac{(2j)!s^{2j}}{((2j)!!)^2}x^{k-2j} =$$
$$= \sum_{j=0}^{+\infty}\left(\sum_{k=2j}^{+\infty}a_k\frac{x^{k-2j}}{(k-2j)!}\right)\frac{s^{2j}}{2^{2j}(j!)^2} =$$
$$= \sum_{j=0}^{+\infty}\left(\sum_{l=0}^{+\infty}a_{l+2j}\frac{x^l}{l!}\right)\left(\frac{1}{j!}\left(\frac{s}{2}\right)^j\right)^2 =$$
$$= \sum_{j=0}^{+\infty}\left(D_x^{2j}F(x)\right)\left(\frac{1}{j!}\left(\frac{s}{2}\right)^j\right)^2.$$

End of proof.

Theorem 3. If a periodic $F(x)$ with a period $2l$ can be expanded into a Fourier series:

$$F(x) = \frac{a_0}{2} + \sum_{n=1}^{+\infty}\left(a_n\cos\frac{n\pi x}{l} + b_n\sin\frac{n\pi x}{l}\right), \qquad (9)$$

then its time average can be expressed in the following form:

$$H_s F(x) =$$
$$= \frac{a_0}{2} + \sum_{n=1}^{+\infty}\left(a_n\cos\frac{n\pi x}{l} + b_n\sin\frac{n\pi x}{l}\right)J_0\left(\frac{n\pi s}{l}\right). \qquad (10)$$

Proof

It is clear that $D_x^{2j}\sin x = (-1)^j\sin x$ and $D_x^{2j}\cos x = (-1)^j\cos x$. But zero order Bessel function of the first kind can be expressed in the following form [2]:

$$J_0(s) = \sum_{j=0}^{+\infty}(-1)^j\left(\frac{1}{j!}\left(\frac{s}{2}\right)^j\right)^2. \qquad (11)$$

Then, Theorem 2 yields:

$$\begin{aligned} H_s \sin x &= J_0(s)\sin x; \\ H_s \cos x &= J_0(s)\cos x. \end{aligned} \qquad (12)$$

Analogously,

$$\begin{aligned} H_s \sin \omega x &= J_0(\omega s)\sin \omega x; \\ H_s \cos \omega x &= J_0(\omega s)\cos \omega x, \end{aligned} \qquad (13)$$

because $D_x^{2j}\sin\omega x = \omega^{2j}(-1)^j\sin\omega x$. Also, it is clear that $H_s const = const$ and $J_0(0)=1$. Finally,

$$H_s F(x) =$$
$$= H_s\frac{a_0}{2} + \sum_{n=1}^{+\infty}\left(a_n H_s\cos\frac{n\pi x}{l} + b_n H_s\sin\frac{n\pi x}{l}\right) = \qquad (14)$$
$$= \frac{a_0}{2} + \sum_{n=1}^{+\infty}\left(a_n\cos\frac{n\pi x}{l} + b_n\sin\frac{n\pi x}{l}\right)J_0\left(\frac{n\pi s}{l}\right).$$

End of proof.

Corollary 2. The following equality holds true:

$$H_s\exp(-ix) = \exp(-ix)J_0(s). \qquad (15)$$

Theorem 4. Let $F(x)$ be a greyscale function. Then,

$$H_s F(x) = \Phi^{-1}\left(J_0(sv)\Phi\tilde{F}(x)\right) + \frac{1}{2}, \qquad (16)$$

where H_s is generalised time average operator; $\tilde{F}(x) := F(x) - \frac{1}{2}$.

Proof

Function $\tilde{F}(x)$ can be expressed in a Fourier series in a complex form [7]: $\tilde{F}(x) = \sum_{n=-\infty}^{+\infty} c_n \exp\left(i\frac{n\pi}{l}x\right)$ where $l > 0$ is fixed; $-l \le x \le l$ and $c_n = \frac{1}{2l}\int_{-l}^{l}\tilde{F}(u)\exp\left(-i\frac{n\pi}{l}u\right)du$.

We construct an auxiliary function

$$G_l(x) := \sum_{n=-\infty}^{+\infty} c_n \exp\left(i\frac{n\pi}{l}x\right).$$

It can be noted that $\tilde{F}(x) = G_l(x)$ at $-l < x < l$. Thus,

$$\begin{aligned}
&H_s\tilde{F}(x) = \lim_{l\to+\infty} G_l(x) = \\
&= \lim_{l\to+\infty} H_s \sum_{n=-\infty}^{+\infty}\frac{1}{2l}\int_{-l}^{l}\tilde{F}(u)\exp\left(-i\frac{n\pi}{l}u\right)du \cdot \\
&\cdot \exp\left(i\frac{n\pi}{l}x\right) = \\
&= \lim_{l\to+\infty}\left(\frac{1}{2}\sum_{n=-\infty}^{+\infty}\left(\begin{gathered}\left(\frac{1}{l}\int_{-l}^{l}\tilde{F}(u)\exp\left(-i\frac{n\pi}{l}u\right)du\right)\cdot \\ \cdot J_0\left(\frac{n\pi}{l}s\right)\exp\left(i\frac{n\pi}{l}x\right)\end{gathered}\right)\right) = \\
&= \lim_{\Delta_x\to 0}\frac{1}{2\pi}\sum_{n=-\infty}^{+\infty}\left(\begin{gathered}\left(\int_{-l}^{l}\tilde{F}(u)\exp(-in\Delta_x u)du\right)\cdot \\ \cdot J_0(n\Delta_x s)\exp(in\Delta_x x)\Delta_x\end{gathered}\right) = \\
&= \frac{1}{2\pi}\int_{-\infty}^{+\infty}\left(\left(\int_{-\infty}^{+\infty}\tilde{F}(u)\exp(-ivu)du\right)J_0(vs)\exp(ivx)\right)dv = \\
&= \Phi^{-1}\left(J_0(sv)\Phi\tilde{F}(x)\right).
\end{aligned}$$

As $F(x) = \tilde{F}(x) + \frac{1}{2}$, the statement of the Theorem holds true.

End of proof.

Corollary 3. If two greyscale functions $F_1(x)$ and $F_2(x)$ are different at least at one interval (a,b), where $a < b$, (i.e. $F_1(x) \neq F_2(x)$ for $x \in (a,b)$), then for all $s \ge 0$ exist such intervals of x values where $H_s F_1(x) \neq H_s F_2(x)$.

4. Discussion on the Complexity of the Inverse Problem

Definition 1 limits the set of functions which can be used for time averaging. Moreover, Theorem 4 operates even on a smaller set of functions. Nevertheless, some results (particularly the proposition of Theorem 4) can be replicated for a wider class of functions. We will use the function $\exp(x)$ (which is clearly not a greyscale function) to illustrate this fact.

It is clear that $D_x^n \exp(x) = \exp(x)$. Then, Theorem 2 yields:

$$H_s \exp(x) = \exp(x)\sum_{j=0}^{+\infty}\left(\frac{1}{j!}\left(\frac{s}{2}\right)^j\right)^2 \neq \exp(x)J_0(s).$$

Well, one cannot expect a result analogous to the one produced by Theorem 3 for a non-greyscale function.

On the other hand, not all greyscale functions can be expanded into a Fourier series. Typical example could be a sum of two harmonics with incommensurate frequencies: $F(x) = \frac{1}{2} + \frac{1}{4}\cos x + \frac{1}{4}\sin\sqrt{3}x$. The greyscale function $\tilde{F}(x) = \frac{1}{4}\cos x + \frac{1}{4}\sin\sqrt{3}x$ is not modulus integrable in an infinite interval, but the following equality holds true:

$$\begin{aligned}
&H_s\left(\frac{1}{2} + \frac{1}{4}\cos x + \frac{1}{4}\sin\sqrt{3}x\right) = \\
&= \frac{1}{2} + \frac{1}{4}J_0(s)\cos x + \frac{1}{4}J_0\left(\sqrt{3}s\right)\sin\sqrt{3}x.
\end{aligned}$$

Time averaging of a greyscale function $F(x)$ (one-dimensional image which oscillates harmonically in time) produces greyscale blur. That blur can be characterised as a convolution between the original image (function $F(x)$) and the point spread function [2] which characterises the distribution of deflections from the state of equilibrium in time. The point spread function in our notation is represented as function $p(x)$ which determines the distribution of the random variable which defines the displacements from the state of equilibrium during the process of time averaging (Theorem 4).

Eq. (16) is an important result explaining the fact that time averaged greyscale images of harmonically oscillating objects can be expressed in operator format. We will show that the kernel of the integral transformation in Eq. (16) is irregular and therefore the inverse problem is ill-posed.

The formulation of the inverse problem follows from Eq. (16). Unfortunately, this problem is ill-posed as it involves calculation of the following expression:

$$\frac{1}{J_0(sv)}\Phi\left(H_s F(x)-\frac{1}{2}\right). \qquad (17)$$

Zero order Bessel function of the first kind $J_0(s)$ has multiple roots (Fig. 1). Therefore there exist multiple divisions by zero in Eq. 17 what makes this inverse problem ill-posed. This property is illustrated by Fig. 2 where a number of interference fringes can be observed at increasing amplitude *s*.

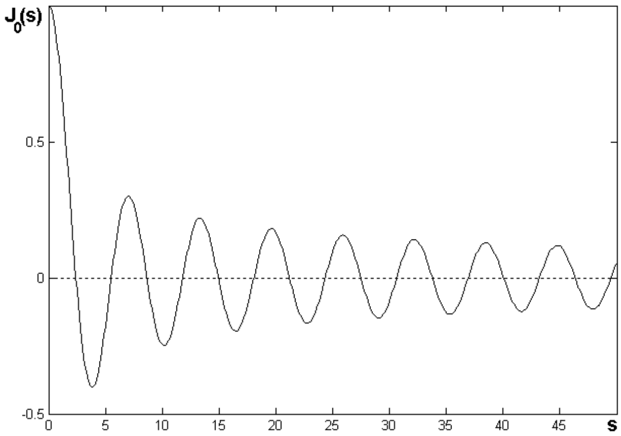

Fig. 1. Zero order Bessel function of the first type

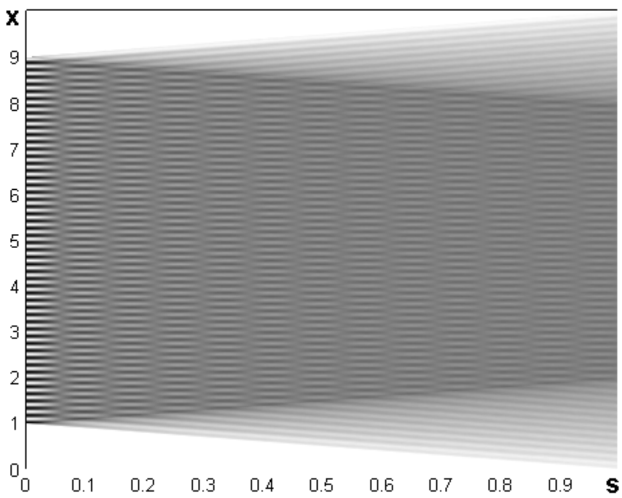

Fig. 2. Interference fringes produced by time averaging of a harmonic grating at increasing amplitudes *s*

Classical, harmonic moiré grating $F(x)=\frac{1}{2}+\frac{1}{2}\cos\left(\frac{2\pi}{\lambda}x\right)$ is used for construction of time averaged image in Fig. 2 ($\lambda = 20$; the background colour is white). This is an elegant illustration of the formation of time averaged fringes:

$$\frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(\frac{1}{2}+\frac{1}{2}\cos\left(\frac{2\pi}{\lambda}(x+s\sin t)\right)\right)dt =$$
$$=\frac{1}{2}+\frac{1}{2}\cos\left(\frac{2\pi}{\lambda}x\right)\lim_{T\to+\infty}\frac{1}{T}\int_0^T\cos\left(\frac{2\pi}{\lambda}s\sin t\right)dt = \qquad (18)$$
$$=\frac{1}{2}+\frac{1}{2}J_0\left(\frac{2\pi}{\lambda}s\right)\cos\left(\frac{2\pi}{\lambda}x\right).$$

The intensity of illumination becomes equal to 0.5 when $\frac{2\pi}{\lambda}s$ coincides with a root of zero order Bessel function of the first type, and a time averaged fringe is formed around this value of *s*.

It is clear that many other different greyscale functions can be used for time averaging applications. Some examples are presented in Fig. 3 and Fig. 4 where time averaged images of a step function and a moiré grating with variable pitch illustrate the complexity of the fringe formation.

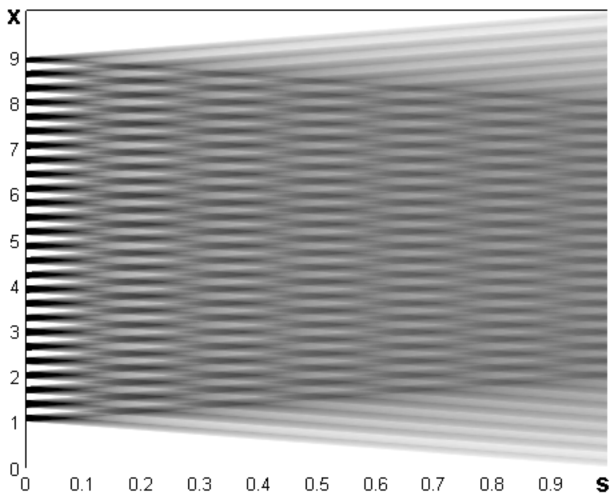

Fig. 3. Time average image of a step function

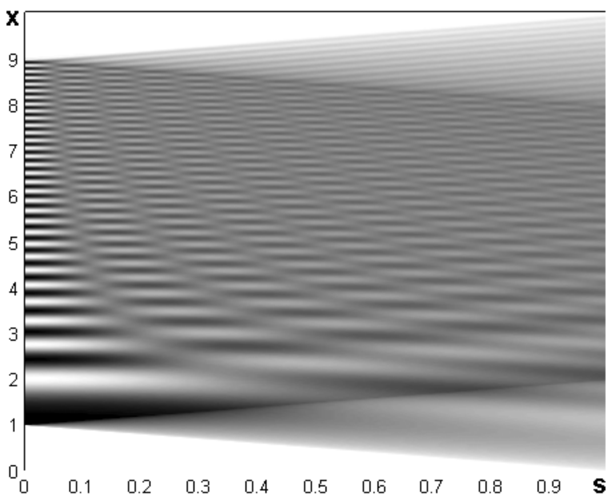

Fig. 4. Time average image of a grating with variable pitch

5. Generalisations for Gaussian and Uniform Distributions

So far, we have analysed harmonic oscillations around the state of equilibrium. Derived analytical relationships describing the process of time averaging for arcsine distribution (harmonic oscillations) can be generalised for Gaussian or uniform distributions. In fact, Lemma 3 allows construction of relationships for different distributions.

For arcsine distribution $E\zeta_s^{2k+1} = 0$; $E\zeta_s^{2k} = \frac{(2k-1)!!}{(2k)!!} s^{2k}$ (Lemma 1). Then, it was shown that $H_s\tilde{F}(x) = \Phi^{-1}\left(J_0(sv)\Phi\tilde{F}(x)\right)$ (Theorem 4). It is well known [8] that time averaged image can be interpreted as a convolution between the original greyscale function and the point spread function describing the motion of the registered object (or camera) – Theorem 1 illustrates this fact. We go even further and express the process of time averaging in operator format which enables clear interpretation of the inverse problem.

Keeping in mind that camera motion is a common factor in experimental optical analysis of dynamical systems, similar results for uniform distribution would be of a high interest.

Uniform density function for a random variable ζ_u is:

$$p_u(x) := (\mathbf{1}(x+u) - \mathbf{1}(x-u))\frac{1}{2u}, \tag{19}$$

where the measured system is initially displaced by $-u$ from the state of equilibrium and then continuously moves with constant velocity until the displacement from the equilibrium is u. Now, $E\zeta_u^{2k+1} = 0$; $E\zeta_u^{2k} = \frac{1}{2k+1}u^{2k}$; $\Phi p_u(x) = \frac{\sin vu}{vu}$;

$$H_u\tilde{F}(x) = \sum_{k=0}^{+\infty}\left(D_x^{2k}\tilde{F}(x)\right)\frac{u^{2k}}{(2k+1)!}, \tag{20}$$

where

$$H_u\tilde{F}(x) := \int_0^1 \tilde{F}(x + u(2y-1))dy. \tag{21}$$

Finally,

$$H_u\tilde{F}(x) = \Phi^{-1}\left(\frac{\sin vu}{vu}\Phi\tilde{F}(x)\right). \tag{22}$$

It can be noted that the kernel of the operator transform in eq. (21) is also ill posed because the roots of sine function prevent exact reconstruction of the original image $\tilde{F}(x)$. In general, the necessity to calculate $\frac{1}{\hat{p}(sv)}\Phi H_s\tilde{F}(x)$ impedes the solution of the inverse problem.

Analogous relationships can be derived for Gaussian distribution which characterises the background noise as a common factor in experimental optical analysis of dynamical systems.

Gaussian density function of a random variable ζ_{σ^2} is:

$$p_{\sigma^2}(x) := \frac{1}{\sqrt{2\pi}\sigma}\exp\left(-\frac{x^2}{2\sigma^2}\right), \tag{23}$$

where σ^2 is dispersion; $\Phi p_{\sigma^2}(x) = \exp\left(-\frac{v^2\sigma^2}{2}\right)$, because $E\zeta_{\sigma^2}^{2k} = (2k-1)!!\sigma^{2k}$.

It can be noted that time averaging operator is now defined slightly different compared to Definition 2:

$$H_{\sigma^2}\tilde{F}(x) := \lim_{T\to+\infty}\frac{1}{T}\int_0^T \tilde{F}\left(x + G_{\sigma^2}^{-1}(t)\right)dt, \tag{24}$$

where $G_{\sigma^2}^{-1}(t)$ is the inverse of Gaussian distribution function $G_{\sigma^2}(x) = \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{x}\exp\left(-\frac{u^2}{2\sigma^2}\right)du$.

Finally,

$$H_{\sigma^2}\tilde{F}(x) = \Phi^{-1}\left(\exp\left(-\frac{v^2\sigma^2}{2}\right)\Phi\tilde{F}(x)\right). \tag{25}$$

It can be noted that eq. (25) is also ill-posed because $\lim_{x\to+\infty}\exp\left(\frac{v^2x^2}{2}\right) = +\infty$.

Such considerations can be formally generalised for general distributions.

6. Two Properties of Greyscale Averaging

If the displacement from the state of equilibrium is governed by a random variable ζ_m density function of which is $p_m(x)$ ($p_m(x)$ is a symmetric real function), time average operator can be denoted as:

$$H\left(\tilde{F}(x)\middle| p_m(x)\right). \tag{26}$$

Then, from Theorem 4 it follows that:

$$H\left(\tilde{F}(x)\middle|p_m(x)\right)=\int_{-\infty}^{+\infty}\tilde{F}(x+y)p_m(y)dy= \\ =\Phi^{-1}\left(\hat{p}_m(sv)\Phi\tilde{F}(x)\right), \tag{27}$$

where $\hat{p}_m(z)$ is the Fourier transform of $p_m(x)$. Then the following equalities hold true:

(i) Sequential time averaging.

$$\begin{aligned}&H\left(H\left(\tilde{F}(x)\middle|p_{m_1}(x)\right)\middle|p_{m_2}(x)\right)=\\&=\Phi^{-1}\left(\hat{p}_{m_2}(m_2v)\Phi\Phi^{-1}\left(\hat{p}_{m_1}(m_1v)\Phi\tilde{F}(x)\right)\right)=\\&=\Phi^{-1}\left(\hat{p}_{m_2}(m_2v)\hat{p}_{m_1}(m_1v)\Phi\tilde{F}(x)\right)=\\&=\int_{-\infty}^{+\infty}\tilde{F}(x+y)\left(p_{m_1}(y)*p_{m_2}(y)\right)dy=\\&=H\left(\tilde{F}(x)\middle|p_{m_1}(x)*p_{m_2}(x)\right)=\\&=E\tilde{F}\left(x+\left(\zeta_{m_1}+\zeta_{m_2}\right)\right).\end{aligned} \tag{28}$$

(ii) Composite averaging.

$$\begin{aligned}&H\left(\tilde{F}(x)\middle|ap_{m_1}(x)+bp_{m_2}(x)\right)=\\&=\Phi^{-1}\left(\left(a\hat{p}_{m_1}(x)+b\hat{p}_{m_2}(x)\right)\Phi\tilde{F}(x)\right)=\\&=\Phi^{-1}\left(a\hat{p}_{m_1}(x)\Phi\tilde{F}(x)\right)+\Phi^{-1}\left(b\hat{p}_{m_2}(x)\Phi\tilde{F}(x)\right)=\\&=aH\left(\tilde{F}(x)\middle|p_{m_1}(x)\right)+bH\left(\tilde{F}(x)\middle|p_{m_2}(x)\right)=\\&=aE\tilde{F}\left(x+\zeta_{m_1}\right)+bE\tilde{F}\left(x+\zeta_{m_2}\right).\end{aligned} \tag{29}$$

Particularly for Gaussian distribution, sequential time averaging can be expressed by single averaging (this property does not hold true neither for arcsine, nor uniform distributions). Really, if $\zeta_{\sigma_1^2}\sim N\left(0,\sigma_1^2\right)$; $\zeta_{\sigma_2^2}\sim N\left(0,\sigma_2^2\right)$, then:

$$E\left(\zeta_{\sigma_1^2}+\zeta_{\sigma_2^2}\right)^{2k+1}=E\sum_{j=0}^{2k+1}\binom{2k+1}{j}\zeta_{\sigma_1^2}^{j}\zeta_{\sigma_2^2}^{2k+1-j}= \\ =\sum_{j=0}^{+\infty}\binom{2k+1}{j}\left(E\zeta_{\sigma_1^2}^{j}\right)\left(E\zeta_{\sigma_2^2}^{2k+1-j}\right)=0, \tag{30}$$

as j or $2k+1-j$ is odd and thus either $E\zeta_{\sigma_1^2}^{j}$ or $E\zeta_{\sigma_2^2}^{2k+1-j}$ is equal to zero.

On the other hand,

$$\begin{aligned}&E\left(\zeta_{\sigma_1^2}+\zeta_{\sigma_2^2}\right)^{2k}=\\&=\sum_{j=0}^{k}\binom{2k}{2j}(2j-1)!!\sigma_1^{2j}(2k-2j-1)!!\sigma_s^{2k-2j}=\\&=\sum_{j=0}^{k}\frac{(2k)!(2j-1)!!(2(k-j)-1)!!}{(2j)!(2(k-j))!}\left(\sigma_1^2\right)^j\left(\sigma_2^2\right)^{k-j}=\\&=(2k)!\sum_{j=0}^{k}\frac{\left(\sigma_1^2\right)^j}{(2j)!!}\frac{\left(\sigma_2^2\right)^{k-j}}{(2(k-j))!!}=\\&=\frac{(2k)!}{2^k k!}\sum_{j=0}^{+\infty}\frac{k!}{j!(k-j)!}\left(\sigma_1^2\right)^j\left(\sigma_2^2\right)^{k-j}=\\&=\frac{(2k)!}{(2k)!!}\sum_{j=0}^{+\infty}\binom{k}{j}\left(\sigma_1^2\right)^j\left(\sigma_2^2\right)^{k-j}=\\&=(2k-1)!!\left(\sigma_1^2+\sigma_2^2\right)^k.\end{aligned} \tag{31}$$

Thus, a Gaussian distribution $N\left(0,\sigma_1^2+\sigma_2^2\right)$ is produced:

$$H\left(H\left(\tilde{F}(x)\middle|p_{\sigma_1^2}(x)\right)\middle|p_{\sigma_2^2}(x)\right)= \\ =H\left(\tilde{F}(x)\middle|p_{\sigma_1^2+\sigma_2^2}(x)\right). \tag{32}$$

7. Computational Examples

We will use several computational examples to illustrate sequential averaging for different distributions.

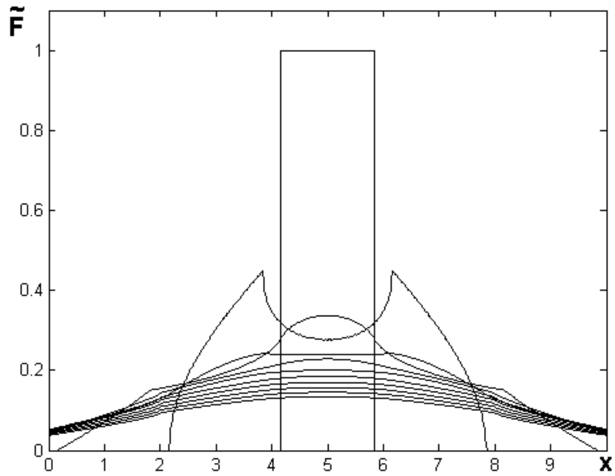

Fig. 5. Sequential averaging of a step function; arcsine distribution; s = 2

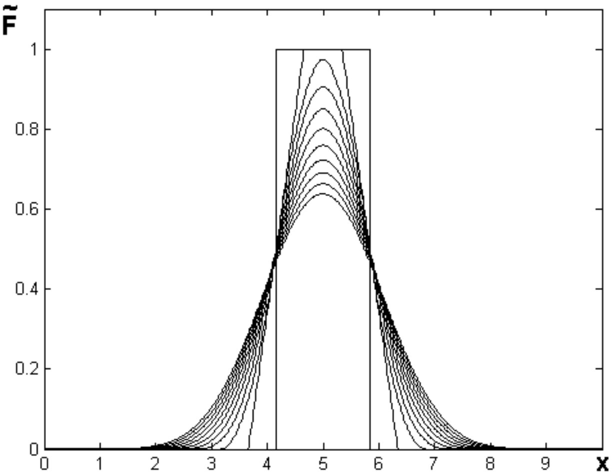

Fig. 6. Sequential averaging of a step function; uniform distribution; $\sigma = 1$

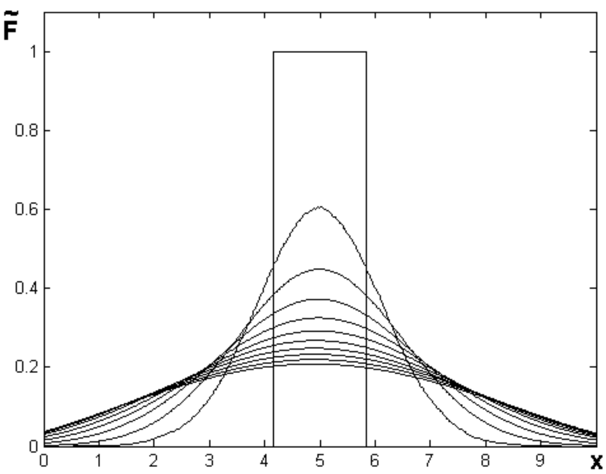

Fig. 7. Sequential averaging of a step function; Gaussian distribution; $u = 0.5$

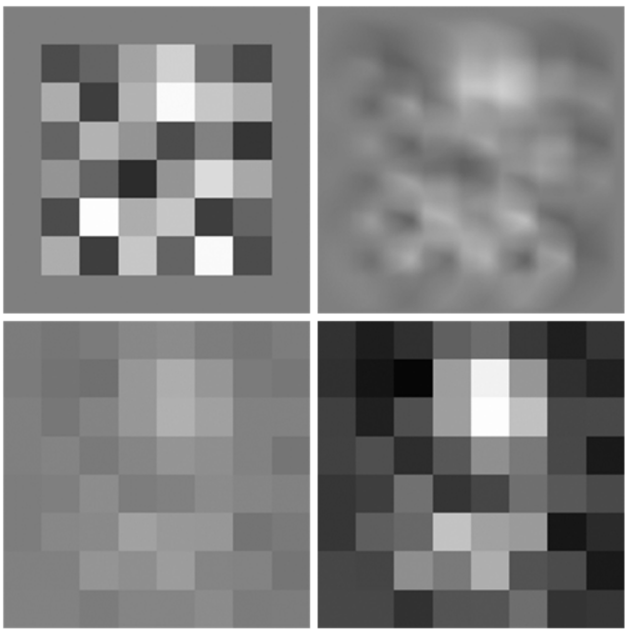

Fig. 8. Illustration of hash algorithm based on time averaging of greyscale images

Figure 5 is a clear illustration that sequential averaging with arcsine distribution cannot be expressed in a single averaging with arcsine distribution. This property can be exploited in cryptographic applications as additional information security factor [6]. The same property holds also for Figure 6, but the kernel of averaging operator with arcsine distribution has an infinite number of singular points and therefore is better applicable for the generation of hash functions [6].

Fig. 8 illustrates the applicability of time averaging techniques (two-dimensional averaging) for construction of one-way hash functions. The original data (left top image in Fig. 8) is blurred by arcsine and Gaussian distribution (right top image). Then, averaged greyscale intensities are reconstructed at the centres of appropriate pixels (left bottom image) and finally stretched to min-max levels (right bottom image) what constitutes the hash value of the illustrated function.

8. Conclusions

Digital image formation during the process of time averaging is expressed in the operator format. Such interpretation helps to explain the complexity of the inverse problem and enables the justification of applicability of arcsine distribution for cryptographic applications.

References

[1] Namboodiri V. P., S. Chaudhuri. On defocus, diffusion and depth estimation. Pattern Recognition Letters, Vol.28(3) (2007), p. 311-319.

[2] Ragulskis M., Maskeliūnas R., Saunorienė L. Identification of in-plane vibrations using time average stochastic moire, Experimental Techniques, 29 (2004), p. 41–45.

[3] Huntley J. M.. Automated fringe pattern analysis in experimental mechanics: a review. Journal of Strain Analysis, 33 (1998), p.105-125.

[4] Kobayashi A. S.. Handbook on Experimental Mechanics, Second Edition. Bethel, SEM, 1993.

[5] Patorski K., Kujawinska M.. Handbook on the Moire Fringe Technique. Oxford, Elsevier, 1993.

[6] Ragulskis M., Navickas Z.. Hash function construction based on time average moiré. Discrete and Continuous Dynamical Systems - Series B, American Institute of Mathematical Sciences, 8(4), (2007) p. 1007-1020.

[7] Jeresnov M., Kiselev A., Makarenko G., Shikin E.. Mathematical Analysis for Engineers. Vol. 2. Moscow, Mir, 1990.

[8] Janssen A. J. E. M. Extended Nijboer-Zernike approach for the computation of optical point-spread functions. Journal of Optical Society of America. A 19, (2002) p. 849-857.